RESEARCH ARTICLE

A new modified Lehmann type – II G class of distributions: exponential distribution with theory, simulation, and applications to engineering sector [version 1; peer review: 1 approved with reservations, 1 not approved]

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Abstract

Background: Modeling against non-normal data a challenge for theoretical and applied scientists to choose a lifetime model and expect to perform optimally against experimental, reliability engineering, hydrology, ecology, and agriculture sciences, phenomena.

Method: We have introduced a new G class that generates relatively more flexible models to its baseline and we refer to it as the new modified Lehmann Type – II (ML-II) G class of distributions. A list of new members of ML-II-G class is developed and as a sub-model the exponential distribution, known as the ML-II-Exp distribution is considered for further discussion. Several mathematical and reliability characters along with explicit expressions for moments, quantile function, and order statistics are derived and discussed in detail. Furthermore, plots of density and hazard rate functions are sketched out over the certain choices of the parametric values. For the estimation of the model parameters, we utilized the method of maximum likelihood estimation.

Results: The applicability of the ML-II-G class is evaluated via ML-II-Exp distribution. ML-II-Exp distribution is modeled to four suitable lifetime datasets and the results are compared with the well-known competing models. Some well recognized goodness-of-fit including -Log-likelihood (-LL), Anderson-Darling (A*), Cramer-Von Mises (W*), and Kolmogorov-Smirnov (K-S) test statistics are considered for the selection of a better fit model.

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Conclusion: The minimum value of the goodness-of-fit is the criteria of a better fit model that the ML-II-Exp distribution perfectly satisfies. Hence, we affirm that the ML-II-Exp distribution is a better fit model than its competitors.

Keywords

Lehmann Type – II Distribution; Exponential Distribution; Hazard Rate Function; Moments; Rényi Entropy; Maximum Likelihood Estimation, Simulation.

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Introduction

Over the past two decades, the growing attention of researchers towards the development of new G families has explored the remarkable characteristics of baseline models. New models open new horizons for theoretical and applied researchers to address real-world problems, to proficiently and adequately fit them to asymmetric and complex random phenomena. Accordingly, several classes of distributions have been developed and discussed in the literature. For more details, we encourage the reader to see the credible work of some notable scientists including Marshall and Olkin¹ $[G(x)/(G(x) + \alpha G(x))]$, who developed a new technique to generate models; Eugene *et al.*,² who proposed beta generated-G class $[G_{\text{beta}}(x)]$; the quadratic rank transmutation map technique introduced by Shaw and Buckley³ $[(1 + \lambda)G(x) - \lambda G^2(x)]$; the Kumaraswamy generalized-G class proposed by Cordeiro and Castro⁴ $[1 - [1 - G^{\alpha}(x)]^{\beta}]$; the gamma-G class proposed by Ristic and Balakrishnan⁵ $[-\log G(x)]$; the exponentiated generalized (EG) class proposed by Cordeiro *et al.*⁶ $[(1 - [1 - G(x)]^{\alpha})^{\beta}]$; the T-X family proposed by Alzaatreh *et al.*⁷ $[1 - R(W[G(x)])]$; the Weibull-G family proposed by Bourguignon *et al.*⁸ $[1 - \exp(-\alpha[G(x)/\overline{G}(x)]^{\beta})]$; the beta Marshall-Olkin-G class proposed by Alizadeh *et al.*⁹ $[I_{\text{Marshall-Olkin}}(a, b)]$; the logistic-X family proposed by Tahir *et al.*¹⁰ $[[1 + [-\log \overline{G}(x)]^{-\alpha}]^{-1}]$; Kumar *et al.*¹¹ proposed DUS transformation $[(e^{G(x)} - 1)/(e - 1)]$; Mahdavi and Kundu¹² proposed Alpha transformation $[(\alpha^{G(x)} - 1)/(\alpha - 1)]$; Elbatal *et al.*¹³ proposed a new Alpha power transformed family of distributions $[G(x)\alpha^{G(x)}/\alpha]$; and recently, Ijaz *et al.*¹⁴ proposed a Gull alpha power Weibull family $[\alpha G(x)/\alpha^{G(x)}]$; among others.

This paper is organised into the following sections. A new modified Lehmann Type-II (ML-II) G class of distributions accompanied by a table of some special models is proposed and developed in *A new modified Lehmann Type-II-G Class of distributions*. A special model of ML-II-G class, known as a modified Lehmann Type-II exponential (ML-II-Exp) distribution, along with its mathematical and reliability measures are derived and discussed in *Mathematical properties*. The method of maximum likelihood estimation is used to estimate the unknown model parameters and develop some simulation results to assess the performance of maximum likelihood estimations (MLEs) in *Inference*. Applications are discussed in *four real data applications* and finally, the conclusion is reported in *Conclusions*.

A new modified Lehmann Type - II-G Class of distributions

Lehmann¹⁵ proposed the Lehmann Type-I (L - I) distribution, which was the simple exponentiated version of any arbitrary baseline model. Accordingly, the first credit goes to Gupta *et al.*¹⁶ because they applied L - I on exponential distribution. The associated cumulative distribution function (CDF) is given by

$$F(x; \alpha, \xi) = G^{\alpha}(x; \xi)$$

Cordeiro *et al.*⁴ deserves to be acknowledged as they use dual transformation that yielded the Lehmann Type - II (L-II) G class of distributions. The associated CDF is given by

$$F(x; \alpha, \xi) = 1 - (1 - G(x; \xi))^{\alpha}$$

The closed-form feature of the L-II distribution assists one to derive and study its numerous properties and in literature, both the approaches (L-I and L-II) have been extensively utilized to study the unexplored characteristics of the classical and modified models.

We develop a new G class, known as a modified Lehmann Type-II (ML-II) G class of distributions. The corresponding CDF is given by

$$F_{\text{ML-II-G}}(x; \alpha, \beta, \xi) = 1 - \left(\frac{1 - G(x; \xi)}{1 - \alpha G(x; \xi)} \right)^{\beta}, x \in \mathbb{R} \quad (1)$$

where $G(x; \xi) \in (0, 1)$ is a CDF of any arbitrary baseline model based on the parametric vector ξ , dependent on $(r \times 1)$ with $-\infty < \alpha < 1$, and $\alpha, \beta > 0$ are the scale and shape parameters, respectively. Let $g(x; \xi) = dG(x; \xi)/dx$ be the density function of any baseline model. The associated probability density function (PDF), hazard rate function (HRF), and quantile function of ML-II-G class of distributions are given by, respectively

$$f_{\text{ML-II-G}}(x; \alpha, \beta, \xi) = \beta(1 - \alpha)g(x; \xi)(1 - G(x; \xi))^{\beta-1}(1 - \alpha G(x; \xi))^{-(1+\beta)} \quad (2)$$

$$h_{\text{ML-II-G}}(x; \alpha, \beta, \xi) = \frac{\beta(1 - \alpha)g(x; \xi)(1 - G(x; \xi))^{\beta-1}(1 - \alpha G(x; \xi))^{-(1+2\beta)}}{(1 - \alpha G(x; \xi))^{\beta} - (1 - G(x; \xi))^{\beta}} \quad (3)$$

$$Q_{ML-II-G}(p; \alpha, \beta, \zeta) = G^{-1} \left(\frac{(1-p)^{1/\beta} - 1}{\alpha(1-p)^{1/\beta} - 1} \right), p \in (0, 1) \quad (4)$$

Now and onward, the modified Lehmann Type-II-G random variable X corresponding to $f_{ML-II-G}(x; \alpha, \beta, \zeta)$ will be denoted by $X \sim ML-II-G(x; \alpha, \beta, \zeta)$.

This study aimed to propose a new G class of distributions that generates more flexible alternative continuous models relative to its parent distribution. From a computational point of view, new models are very simple to interpret. New models offer greater distributional flexibility and can provide a better fit over the complex random phenomena that exclusively arise in engineering sciences. However, to the best of our knowledge, no study has been conducted previously that discusses our new G class, deliberates the unexplained complex random phenomena so well and advances the fit to a diverse range of sophisticated lifetime data.

Linear combination provides a much more informal approach to discuss the CDF and PDF than the conventional integral computation when determining the mathematical properties. For this, binomial expansion is given as follows:

$$(1-z)^\beta = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} z^i, |z| < 1$$

Infinite linear combinations of CDF (1) and PDF (2) of ML-II-G class are given by, respectively

$$F(x; \alpha, \beta, \zeta) = 1 - \sum_{i,j=0}^{\infty} (-1)^{i+j} \alpha^j \binom{\beta}{i} \binom{-\beta}{j} G^{i+j}(x; \alpha, \beta, \zeta)$$

$$F(x; \alpha, \beta, \zeta) = 1 - \sum_{i,j=0}^{\infty} \omega_{ij}^* G^{i+j}(x; \alpha, \beta, \zeta) \quad (5)$$

$$f(x; \alpha, \beta, \zeta) = \beta(1-\alpha) \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{\beta-1}{i} \binom{-\beta-1}{j} G^{i+j}(x; \alpha, \beta, \zeta) g(x; \alpha, \beta, \zeta)$$

$$f(x; \alpha, \beta, \zeta) = \beta(1-\alpha) \sum_{i,j=0}^{\infty} \omega_{ij} G^{i+j}(x; \alpha, \beta, \zeta) g(x; \alpha, \beta, \zeta) \quad (6)$$

The r -th ordinary moment (say μ_r') of X is given by

$$\mu_{r,I}' = \beta(1-\alpha) \sum_{i,j=0}^{\infty} \omega_{ij} I_{r,i+j}(x; \zeta) \quad (7)$$

where coefficient $\omega_{ij}^* = (-1)^{i+j} \alpha^j \binom{\beta}{i} \binom{-\beta}{j}$, $\omega_{ij} = \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{\beta-1}{i} \binom{-\beta-1}{j}$, and $I_{r,i+j}(x; \zeta) = \int_{-\infty}^{\infty} x^r G^{i+j}(x; \zeta) g(x; \zeta) dx$.

Further, by (1), Lehmann Type-II-G class and baseline $G(x)$ model can be traced back at $\alpha=0$, and $\alpha=0, \beta=1$, respectively. The expansions of CDF (5) and PDF (6) provide us with the Exp-G class, which is quite useful for the generalization of models.

Order statistics

In reliability analysis and life testing of a component in quality control, order statistics (OS) and its moments are considered as a noteworthy measure. Let $X_1, X_2, \dots, X_n$ be a random sample of size n that follows the Lehmann Type-II-G class and $X_{(1:n)} < X_{(2:n)} < \dots < X_{(n:n)}$ be the corresponding OS. The random variables $X_{(i)}$, $X_{(1)}$, and $X_{(n)}$ be the i -th, minimum, and maximum OS of X .

The PDF of $X_{(i)}$ is given by

$$f_{(i:n)}(x; \zeta) = \frac{1}{B(i, n-i+1)!} (F(x; \zeta))^{i-1} (1-F(x; \zeta))^{n-i} f(x; \zeta)$$

where $= 1, 2, 3, \dots, n$, $F(x; \xi)$, and $f(x; \xi)$ are the associated CDF (5) with corresponding PDF (6) of the Lehmann Type-II-G family. Using the fact that

$$(1 - F(x; \xi))^{n-i} = \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} F(x; \xi)^m$$

By placing the last expression in $f_{(in)}(x; \xi)$, we get the most refined form of OS PDF and one may determine it by integrating (5–6) and the expression may be given as follows

$$f_{(in)}(x; \xi) = \frac{f(x; \xi)}{B(i, n-i+1)!} \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} F(x; \xi)^{i+m-1}$$

A modified Lehmann Type – II Exponential (ML-II-Exp) distribution (sub-model)

In this section, we introduce a sub-model of ML-II-G class of distributions, known as a ML-II-Exp distribution. For this, we have the CDF and PDF of the exponential distribution $G_{Exp}(x; \theta) = 1 - e^{-\theta x}$, and $g_{Exp}(x; \theta) = \theta e^{-\theta x}$, for $\theta > 0, 0 < x < \infty$, respectively. The associated CDF and corresponding PDF of the ML-II-Exp distribution are obtained by following (1–2) and its analytical expressions are given by respectively:

$$F_{ML-II-Exp}(x; \alpha, \beta, \theta) = 1 - \left(\frac{e^{-\theta x}}{1 - \alpha + \alpha e^{-\theta x}} \right)^\beta, \quad -\infty < \alpha < 1 \quad (8)$$

$$f_{ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{(1 - \alpha)\beta\theta e^{-\theta\beta x}}{(1 - \alpha + \alpha e^{-\theta x})^{\beta+1}} \quad (9)$$

where α is a scale and $\beta, \theta > 0$ are the shape parameters, respectively. By following (1–2), linear representation of CDF and PDF are given as follows, respectively

$$F(x; \alpha, \beta, \theta) = 1 - \sum_{i,j=0}^{\infty} (-1)^j \theta^j (\beta + i)^j \binom{-\beta}{i} \frac{\alpha^j (1 - \alpha)^{-\beta-i}}{j!} x^j \quad (10)$$

$$F(x; \alpha, \beta, \theta) = 1 - \sum_{i,j=0}^{\infty} \zeta_{ij} x^j$$

$$f(x; \alpha, \beta, \theta) = \beta\theta \sum_{i=0}^{\infty} \alpha^i (1 - \alpha)^{-\beta} \binom{-\beta-1}{i} e^{-(i+\beta)\theta x} \quad (11)$$

$$f(x; \alpha, \beta, \theta) = \beta \sum_{i,j=0}^{\infty} (-1)^j (1 - \alpha)^{-\beta-i} \alpha^i \theta^{j+1} \binom{-\beta-1}{i} \frac{(\beta+i)^j}{j!} x^j \quad (12)$$

$$f(x; \alpha, \beta, \theta) = \beta \sum_{i,j=0}^{\infty} \zeta_{ij}^* x^j.$$

where $\zeta_{ij} = (-1)^j \theta^j (\beta + i)^j \binom{-\beta}{i} \frac{\alpha^j (1 - \alpha)^{-\beta-i}}{j!}$, $\zeta_{ij}^* = (-1)^j (1 - \alpha)^{-\beta-i} \alpha^i \theta^{j+1} \binom{-\beta-1}{i} \frac{(\beta+i)^j}{j!}$

Expression in (11) is expected to be quite supportive in the forthcoming computations of various mathematical properties of the ML-II-Exp distribution.

Mathematical properties

Reliability characteristics

One of the imperative roles of probability distribution in reliability engineering is to analyze and predict the life of a component. One may define the reliability function as the probability that a component survives until the time x and analytically it can be written as $R(x) = 1 - F(x)$.

The reliability function of X is given by

$$R_{ML-II-Exp}(x; \alpha, \beta, \theta) = \left(\frac{e^{-\theta x}}{1 - \alpha + \alpha e^{-\theta x}} \right)^\beta \quad (13)$$

In reliability theory, the significant contribution of a function that measures the failure rate of a component in a particular time t is sometimes referred to as the HRF, failure rate function, or the force of mortality, and mathematically it can be written as $h(x) = f(x)/R(x)$.

The HRF of X is given by

$$h_{ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{(1 - \alpha)\beta\theta}{(1 - \alpha + \alpha e^{-\theta x})} \quad (14)$$

Numerous notable reliability measures for the ML-II-Exp distribution can be discussed and derived, such as reverse HRF by $h_r(x; \alpha, \beta, \theta) = f(x; \alpha, \beta, \theta)/R(x; \alpha, \beta, \theta)$, Mills ratio by $M(x; \alpha, \beta, \theta) = R(x; \alpha, \beta, \theta)/f(x; \alpha, \beta, \theta)$, and Odd function by

$$O(x; \alpha, \beta, \theta) = F(x; \alpha, \beta, \theta)/R(x; \alpha, \beta, \theta).$$

Shapes

Different plots of PDF and HRFs of the ML-II-Exp distribution are sketched over the selected and fixed combinations of the model parameters, respectively. Figure 1 (a, b, c) presents the reversed-J, constant, unimodal, and right-skewed shapes of the PDF and Figure 2 (a, b, c) illustrates the decreasing and increasing HRF. However, an increasing HRF with some interesting facts are identified when suddenly spikes arise at the tail end of HRF is unexpectedly detected. Such kinds of trends are often observed in non-stationary time series lifetime phenomena.

Limiting behavior

Here we study the limiting behavior of CDF, PDF, reliability, and HRFs of the ML-II-Exp distribution present in (8), (9), (13), and (14) at $x \rightarrow 0$ and $x \rightarrow \infty$.

Figure 1. Density function plots of ML-II-Exponential distribution.

Figure 2. Hazard rate function plots of ML-II-Exponential distribution.

Table 1. Some special models and corresponding $G(x, \xi)$ and $S(x, \xi)$.

Model	$G(x)$	$S(x)$	ξ
Rayleigh ($x > 0$)	$1 - e^{-\eta x^2}$	$\left(\frac{e^{-\eta x^2}}{1 - \alpha(1 - e^{-\eta x^2})} \right)^\beta$	α, β, η
Gompertz ($x > 0$)	$1 - e^{-\eta(e^{yx} - 1)}$	$\left(\frac{e^{-\eta(e^{yx} - 1)}}{1 - \alpha(1 - e^{-\eta(e^{yx} - 1)})} \right)^\beta$	$\alpha, \beta, \eta, \gamma$
Pareto ($x > m$)	$1 - \left(\frac{m}{x}\right)^\eta$	$\left(\frac{\left(\frac{m}{x}\right)^\eta}{1 - \alpha\left(1 - \left(\frac{m}{x}\right)^\eta\right)} \right)^\beta$	α, β, η
Fréchet ($x > 0$)	$e^{-\eta x^{-\gamma}}$	$\left(\frac{1 - e^{-\eta x^{-\gamma}}}{1 - \alpha e^{-\eta x^{-\gamma}}} \right)^\beta$	$\alpha, \beta, \eta, \gamma$
Burr X ($x > 0$)	$\left(1 - e^{-(\eta x)^2}\right)^\gamma$	$\left(\frac{1 - \left(1 - e^{-(\eta x)^2}\right)^\gamma}{1 - \alpha\left(1 - e^{-(\eta x)^2}\right)^\gamma} \right)^\beta$	$\alpha, \beta, \eta, \gamma$
Weibull ($x > 0$)	$1 - e^{-\eta x^\gamma}$	$\left(\frac{e^{-\eta x^\gamma}}{1 - \alpha(1 - e^{-\eta x^\gamma})} \right)^\beta$	$\alpha, \beta, \eta, \gamma$
Lomax ($x > 0$)	$1 - \left(1 + \frac{x}{\eta}\right)^{-\gamma}$	$\left(\frac{\left(1 + \frac{x}{\eta}\right)^{-\gamma}}{1 - \alpha\left(1 - \left(1 + \frac{x}{\eta}\right)^{-\gamma}\right)} \right)^\beta$	$\alpha, \beta, \eta, \gamma$

Proposition-1: Limiting behavior of CDF, PDF, reliability, and HRFs of the ML-II-Exp distribution at $x \rightarrow 0$ is followed by

$$\begin{aligned}
 F(0; \alpha, \beta, \theta) &\sim 0 \\
 f(0; \alpha, \beta, \theta) &\sim (1 - \alpha)\beta\theta \\
 R(0; \alpha, \beta, \theta) &\sim 1 \\
 h(0; \alpha, \beta, \theta) &\sim (1 - \alpha)\beta\theta
 \end{aligned}$$

Proposition-2: Limiting behavior of CDF, PDF, reliability, and HRFs of the ML-II-Exp distribution at $x \rightarrow \infty$ is followed by

$$\begin{aligned}
 F(\infty; \alpha, \beta, \theta) &\sim 1 \\
 f(\infty; \alpha, \beta, \theta) &\sim 0 \\
 R(\infty; \alpha, \beta, \theta) &\sim 0 \\
 h(\infty; \alpha, \beta, \theta) &\sim 0
 \end{aligned}$$

Limiting behaviors developed in the above expressions may illustrate the effect of parameters on the tail of the ML-II-Exp distribution.

Moments and associated measures

Moments have a remarkable role in the discussion of the distribution theory, to study the significant characteristics of a probability distribution such as mean; variance; skewness, and kurtosis.

Theorem 1: If $X \sim \text{ML-II-Exp}(x; \alpha, \beta, \theta)$, for $x, \beta, \theta > 0$, with $-\infty < \alpha < 1$, then the r -th ordinary moment (say μ_r') of X is given by

$$\mu_r' = \frac{\beta}{\theta^r} \sum_{i=0}^{\infty} \frac{\eta_i}{(\beta + i)^{r+1}} \Gamma(r + 1)$$

$$\text{where } \eta_i = \alpha^i (1 - \alpha)^{-\beta - i} \binom{-\beta - 1}{i}$$

Proof: μ_r' can be written by following (11), as

$$\mu_r' = \beta\theta \sum_{i=0}^{\infty} \alpha^i (1 - \alpha)^{-\beta - i} \binom{-\beta - 1}{i} \int_0^{\infty} x^r e^{-\theta x(\beta + i)} dx \quad (15)$$

Let's suppose $\theta x(\beta + i) = y; x = \frac{y}{\theta(\beta + i)} \Rightarrow dx = \frac{dy}{\theta(\beta + i)}$

limits: as $x \rightarrow 0 \Rightarrow x \rightarrow \infty; y \rightarrow 0 \Rightarrow y \rightarrow \infty$.

By placing the above information in (15), we get

$$\mu_r' = \beta \theta \sum_{i=0}^{\infty} \alpha^i (1-\alpha)^{-\beta-i} \binom{-\beta-1}{i} \int_0^{\infty} \left(\frac{y}{\theta(\beta+i)} \right)^r \frac{e^{-y}}{\theta(\beta+i)} dy$$

by making simple computation on the last expression leads us to the r -th ordinary moment, in terms of the *gamma function* and it is given by

$$\mu_r' = \frac{\beta}{\theta^r} \sum_{i=0}^{\infty} \frac{\eta_i}{(\beta+i)^{r+1}} \Gamma(r+1), r = 1, 2, \dots, n \quad (16)$$

where $\Gamma(\cdot)$ is a gamma function, $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$, $\eta_i = \alpha^i (1-\alpha)^{-\beta-i} \binom{-\beta-1}{i}$, and $-\infty < \alpha < 1$.

The derived expression in (16) may serve a supportive and useful role in the development of several statistical measures. For instance: to deduce the mean of X , place $r = 1$ in (16). For the higher-order ordinary moments of X approximating to 2nd, 3rd, and 4th, and higher, these moments can be formulated by setting $r = 2, 3$, and 4 in (16), respectively. Additionally, to discuss the variability in X , the Fisher index ($F.I = (Var(X)/E(X))$) may play a significant role in this scenario. To derive the negative moments of X , simply substitute r by $-w$ in (16). One may perhaps further determine the well-established statistics such as skewness ($\tau_1 = \mu_3'/\mu_2'^3$), and kurtosis ($\tau_2 = \mu_4'/\mu_2'^2$), of X by integrating (16). A well-established relationship between the central moments (μ_s) and cumulants (K_s) of X may easily be defined by ordinary moments $\mu_s = \sum_{k=0}^s \binom{s}{k} (-1)^k (\mu_1')^s \mu_{s-k}'$. Hence, the first four cumulants can be calculated by $K_1 = \mu_1'$, $K_2 = \mu_2' - \mu_1'^2$, $K_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$, and $K_4 = \mu_4' - 4\mu_3'\mu_1' - 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^4$, etc., respectively.

In **Table 2**, some numerical results of the first eight ordinary moments, ε^2 = variance, τ_1 = skewness, and τ_2 = kurtosis for some chosen parameters are presented in S-I ($\alpha = 0.9, \theta = 0.1, \beta = 1.1$), S-II ($\alpha = 0.9, \theta = 0.5, \beta = 1.5$), S-III ($\alpha = 0.8, \theta = 0.09, \beta = 3.1$), and S-IV ($\alpha = 0.5, \theta = 0.009, \beta = 5.1$).

Residual and reversed residual life functions

The residual life function/conditional survivor function of random variable $R(t) = X + t/X > t$ is the probability that a component whose life says x , survives in an additional interval at $t \geq 0$. Analytically it can be written as

$$S_{R(t)}(x) = \frac{S(x+t)}{S(t)}$$

Table 2. Some numerical results of moments, variance, skewness, and kurtosis.

μ_r'	S-I	S-II	S-III	S-IV
μ_1'	1.0669	1.0953	0.4328	0.3951
μ_2'	2.2184	2.0624	0.3682	0.3119
μ_3'	6.8258	5.3306	0.4638	0.3688
μ_4'	27.8026	17.3849	0.7714	0.5811
μ_5'	141.0179	68.4275	1.5920	1.1437
μ_6'	856.5819	316.0155	3.9204	2.6994
μ_7'	6063.8300	1678.4290	11.2159	7.4292
μ_8'	49031.2300	10095.8100	36.5526	23.3579
ε^2	1.9427	1.7261	0.2578	0.2129
τ_1	2.9596	1.8225	1.9987	2.0879
τ_2	5.0391	3.3552	4.5027	4.7385

The residual life function of X is given by

$$S_{R(t)-ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{e^{-\theta\beta(x+t)}(1 - \alpha + \alpha e^{-\theta t})^\beta}{e^{-\theta\beta t}(1 - \alpha + \alpha e^{-\theta(x+t)})^\beta}, \quad x > 0 \quad (17)$$

with associated CDF

$$F_{R(t)-ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{e^{-\theta\beta t}(1 - \alpha + \alpha e^{-\theta(x+t)})^\beta - e^{-\theta\beta(x+t)}(1 - \alpha + \alpha e^{-\theta t})^\beta}{e^{-\theta\beta t}(1 - \alpha + \alpha e^{-\theta(x+t)})^\beta} \quad (18)$$

PDF and HRF corresponding to (18) are given as follows, respectively

$$f_{R(t)-ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{\beta\theta(1 - \alpha)(1 - \alpha + \alpha e^{-\theta t})^\beta e^{-\beta\theta(x+t) + \theta(x+t) + \beta t\theta}}{(1 - \alpha + \alpha e^{-\theta(x+t)})^\beta (\alpha + (1 - \alpha)e^{\theta(x+t)})} \quad (19)$$

$$h_{R(t)-ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{\beta\theta(1 - \alpha)e^{-\beta\theta(x+t)}}{1 - \alpha + \alpha e^{-\theta(x+t)}} \quad (20)$$

Mean residual life function is given by

$$E(S_R(x)) = \frac{1}{S(t)} \left(\mu'_1 - \int_0^t x f(x) dx \right) - t, \quad t \geq 0$$

Moreover, the reverse residual life can be defined as: $\bar{R}(t) = t - X/X \leq t$.

$$S_{\bar{R}(t)}(x) = \frac{F(t-x)}{F(t)}, \quad t \geq 0$$

The reverse residual life function of X is given by

$$S_{\bar{R}(t)-ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{\left((1 - \alpha + \alpha e^{-\theta(t-x)})^\beta - e^{-\beta\theta(t-x)} \right) (1 - \alpha + \alpha e^{-\theta t})^\beta}{\left((1 - \alpha + \alpha e^{-\theta t})^\beta - e^{-\beta\theta t} \right) (1 - \alpha + \alpha e^{-\theta(t-x)})^\beta} \quad (21)$$

with associated CDF

$$F_{\bar{R}(t)-ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{\left(\left((1 - \alpha + \alpha e^{-\theta t})^\beta - e^{-\beta\theta t} \right) (1 - \alpha + \alpha e^{-\theta(t-x)})^\beta - \left((1 - \alpha + \alpha e^{-\theta(t-x)})^\beta - e^{-\beta\theta(t-x)} \right) (1 - \alpha + \alpha e^{-\theta t})^\beta \right)}{\left((1 - \alpha + \alpha e^{-\theta t})^\beta - e^{-\beta\theta t} \right) (1 - \alpha + \alpha e^{-\theta(t-x)})^\beta} \quad (22)$$

PDF and HRF corresponding to (22) are given as follows, respectively

$$f_{\bar{R}(t)-ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{-\beta\theta(1 - \alpha)(1 - \alpha + \alpha e^{-\theta t})^\beta e^{-\beta\theta(t-x) + \theta(t-x) + \beta t\theta}}{\left(\left((1 - (1 - \alpha + \alpha e^{-\theta t})^\beta) e^{\beta t\theta} \right) (1 - \alpha + \alpha e^{-\theta(t-x)})^\beta \right) (\alpha + (1 - \alpha)e^{\theta(t-x)})} \quad (23)$$

$$h_{\bar{R}(t)-ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{-\beta\theta(1 - \alpha)e^{-\beta\theta(t-x)}}{1 - \alpha + \alpha e^{-\theta(t-x)}} \quad (24)$$

Mean reversed residual life function/mean waiting time is given by

$$E(S_{\bar{R}}(x)) = t - \frac{1}{F(t)} \int_0^t x f(x) dx, t \geq 0$$

One may derive the strong mean inactivity time of X by following

$$M(t) = t^2 - \frac{1}{f(t)} \int_0^t x^2 f(x) dx \text{ for } t \geq 0$$

where $\Phi_r(t) = \int_0^t x^r f(x) dx$, is the r -th incomplete moment, $\mu'_{1t} = \int_0^t x f(x) dx = \frac{\beta}{\theta} \sum_{i=0}^{\infty} \frac{\eta_i}{(\beta+i)^2} \Gamma_i(2)$, and $\mu'_{2t} = \int_0^t x^2 f(x) dx = \frac{\beta}{\theta^2} \sum_{i=0}^{\infty} \frac{\eta_i}{(\beta+i)^3} \Gamma_i(3)$. By following (16), we may derive directly the μ'_{1t} and μ'_{2t} by holding “ $t = l\theta(\beta + i)$ ” as the upper bound. Furthermore, μ'_{1t} and μ'_{2t} are termed as the first and second lower incomplete moments of X , respectively, with $\eta_i = \alpha^i (1 - \alpha)^{-\beta-i} \binom{-\beta-1}{i}$, and $-\infty < \alpha < 1$.

Entropy

When a system is quantified by disorderedness, randomness, diversity, or uncertainty, in general, it is known as entropy.

Rényi¹⁷ entropy of X is given by

$$H_{\zeta}(X) = \frac{1}{1-\zeta} \log \int_0^{\infty} f^{\zeta}(x) dx, \zeta > 0 \text{ and } \zeta \neq 1 \quad (25)$$

By following (9), we simplify $f(x)$ in terms of $f^{\zeta}(x)$, we get

$$H_{\zeta-ML-II-Exp}(X) = ((1-\alpha)\beta\theta)^{\zeta} (1-\alpha)^{-\zeta(\beta+1)} e^{-\beta\theta\zeta x} \left(1 + \frac{\alpha}{1-\alpha} e^{-\theta x}\right)^{-\zeta(\beta+1)}$$

by placing the above expression in (25), we get

$$\begin{aligned} H_{\zeta-ML-II-Exp}(X) &= \frac{1}{1-\zeta} \log \left(((1-\alpha)\beta\theta)^{\zeta} (1-\alpha)^{-\zeta(\beta+1)} \int_0^{\infty} e^{-\beta\theta\zeta x} \left(1 + \frac{\alpha}{1-\alpha} e^{-\theta x}\right)^{-\zeta(\beta+1)} dx \right) \\ H_{\zeta-ML-II-Exp}(X) &= \frac{1}{1-\zeta} \log \left((\beta\theta)^{\zeta} (1-\alpha)^{-\zeta\beta} \sum_{j=0}^{\infty} \binom{-\zeta(\beta+1)}{j} \left(\frac{\alpha}{1-\alpha}\right)^j \int_0^{\infty} e^{-\theta\zeta(\beta+1)x} dx \right) \end{aligned}$$

hence, solving simple mathematics on the previous equation leads us to the most simplified form of the Rényi entropy of X and it is given by

$$H_{\zeta-ML-II-Exp}(X) = \frac{1}{1-\zeta} \log \left(\tau_{\zeta} \sum_{j=0}^{\infty} v_j \right)$$

where $v_j = \left(\frac{\alpha}{1-\alpha}\right)^j \binom{-\zeta(\beta+1)}{j}$, $\tau_{\zeta} = \frac{1}{\theta\zeta(\beta+1)}$, $-\infty < \alpha < 1$.

Quantile function, and mode

The q^{th} quantile function of ML-II-Exp distribution is obtained by inverting the CDF. Quantile function is defined as $q = F(x_q) = P(X \leq x_q)$, $q \in (0, 1)$.

The quantile function of X is given by

$$x_{q-ML-II-Exp} = \frac{1}{\theta} \log \left(\frac{1 - \alpha(1-q)^{\frac{1}{\beta}}}{(1-\alpha)(1-q)^{\frac{1}{\beta}}} \right) \quad (26)$$

To obtain the 1st quartile, median and 3rd quartile of X , place $q = 0.25, 0.5$, and 0.75 respectively in (26). Henceforth, to generate random numbers, one may assume that the CDF in (8) follows the uniform distribution $u = U(0, 1)$.

The modal value of X is calculated by following the constraint $f'(x; \alpha, \beta, \theta) = \frac{df(x; \alpha, \beta, \theta)}{dx} = 0$. For convenience, $f(x; \alpha, \beta, \theta)$ can be rewritten as

$$f_{ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{(1-\alpha)\beta\theta e^{-\beta\theta x}}{(1-\alpha + \alpha e^{-\theta x})^{\beta+1}}$$

The simplified form of $f'(x; \alpha, \beta, \theta)$ is given by

$$f'_{ML-II-Exp}(x; \alpha, \beta, \theta) = \frac{\alpha e^{-\theta x} - (1-\alpha)\beta}{(1-\alpha + \alpha e^{-\theta x})^{\beta+1}}$$

Hence, solving simple algebra on the previous equation may provide us with the most suitable form of the mode of X in support of $f'(x; \alpha, \beta, \theta) = 0$ and it is given by

$$\hat{x}_{ML-II-Exp} = \frac{1}{\theta} \log \left(\frac{\alpha}{\beta(1-\alpha)} \right), -\infty < \alpha < 1.$$

Stress – strength reliability

Let X_1 and X_2 be defined to discuss the strength and stress of a component, respectively, followed by the same uni-variate family of distributions, which will work in order if $X_2 < X_1$. To discuss the reliability (say R) of X , it is given by $R = P(X_2 < X_1)$.

Theorem 2: Let $X_1 \sim ML-II-Exp(x; \alpha, \beta_1, \theta)$ and $X_2 \sim ML-II-Exp(x; \alpha, \beta_2, \theta)$ be independent random variables following the $ML-II-Exp$ distribution; then the reliability is given by

$$R = \frac{\beta_1}{\beta_1 + \beta_2}$$

Proof: Reliability (R) is defined as

$$R = P(X_2 < X_1) = \int_0^{\infty} f_1(x) F_2(x) dx.$$

R of X can be written by following (9), as

$$R_{ML-II-Exp} = \int_0^{\infty} \left(\frac{(1-\alpha)\beta_1\theta e^{-\theta\beta_1 x}}{(1-\alpha + \alpha e^{-\theta x})^{\beta_1+1}} \right) \left(1 - \left(\frac{e^{-\theta x}}{1-\alpha + \alpha e^{-\theta x}} \right)^{\beta_2} \right) dx, \quad (27)$$

Let's suppose $t = \left(\frac{e^{-\theta x}}{1-\alpha + \alpha e^{-\theta x}} \right)^{\beta_1} \Rightarrow -dt = \frac{(1-\alpha)\beta_1\theta e^{-\theta\beta_1 x}}{(1-\alpha + \alpha e^{-\theta x})^{\beta_1+1}} dx$; and $\frac{\beta_2}{t^{\frac{\beta_2}{\beta_1}}} = \left(\frac{e^{-\theta x}}{1-\alpha + \alpha e^{-\theta x}} \right)^{\beta_2}$

limits: as $x \rightarrow 0 \Rightarrow t \rightarrow 1$; $x \rightarrow \infty \Rightarrow t \rightarrow 0$.

By placing the above information in (27), we have

$$R_{ML-II-Exp} = \int_0^1 \left(1 - t^{\frac{\beta_2}{\beta_1}} \right) dt,$$

Hence, the simple computation of the above expression provides us with the reduced form of R in terms of β_1 and β_2 , as we presume that the R is a function of β_1 with increasing behavior and it is given by

$$R_{ML-II-Exp} = \frac{\beta_2}{\beta_1 + \beta_2}$$

Order statistics

In reliability analysis and life testing of a component in quality control, order statistics OS and its moments are considered as a noteworthy measure. Let $X_1, X_2, \dots, X_n$ be a random sample of size n following the ML-II-Exp distribution and $X_{(1:n)} < X_{(2:n)} < \dots < X_{(n:n)}$ be the corresponding OS. The random variables $X_{(i)}$, $X_{(1)}$, and $X_{(n)}$ be the i -th, *minimum*, and *maximum* OS of X .

The PDF of $X_{(i)}$ is given by

$$f_{(i:n)}(x) = \frac{1}{B(i, n-i+1)!} (F(x))^{i-1} (1-F(x))^{n-i} f(x), i=1, 2, 3, \dots, n$$

By following (8) and (9), the PDF of $X_{(i)}$ takes the form

$$f_{(i:n)-ML-II-Exp}(x; \alpha, \beta, \theta) = \left(\frac{\frac{1}{B(i, n-i+1)!} \left(1 - \left(\frac{e^{-\theta x}}{1-\alpha+ae^{-\theta x}} \right)^\beta \right)^{i-1}}{\left(\left(\frac{e^{-\theta x}}{1-\alpha+ae^{-\theta x}} \right)^\beta \right)^{n-i} \frac{(1-\alpha)\beta\theta e^{-\theta\beta x}}{(1-\alpha+ae^{-\theta x})^{\beta+1}}} \right) \quad (28)$$

by utilizing some the techniques of binomial expansion (mentioned in the new modified Lehmann Type – II–G Class of distributions) to simplify (28), we get the reduced form of $f_{(i:n)}(x; \alpha, \beta, \theta)$ and it is given as follows

$$f_{(i:n)-ML-II-Exp}(x) = \frac{\beta\theta}{B(i, n-i+1)!} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \eta_{jk} e^{-\theta x v} \quad (29)$$

and we determine the linear representation of (29) and it can be written as

$$f_{(i:n)ML-II-Exp}(x) = \frac{\beta}{B(i, n-i+1)!} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \nabla_{jkl} x^l \quad (30)$$

Indeed, (29) has a supportive role in the calculation of r -th moment OS and hereafter, straightforward computation of (29) leads us to the r -th moment OS of X and it is given as follows

$$\mu_{OS-ML-II-Exp}^r = \frac{\beta}{B(i, n-i+1)!} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\eta_{jk}}{(\theta v)^{r+1}} \Gamma(r+1) \quad (31)$$

where $\eta_{jk} = (-1)^j \binom{i-1}{j} \binom{\lambda}{k} \alpha^k (1-\alpha)^{\lambda-k}$, $\lambda = -\beta(n-i+j+1) + 1$, $v = \beta\theta(n-i+j+1)$

$$\nabla_{jkl} = (-1)^{j+k} \binom{i-1}{j} \binom{\lambda}{k} v^l / l!$$

The CDF of $X_{(i)}$ is given by

$$F_{(i:n)}(x) = \sum_{r=i}^n \binom{n}{r} (F(x))^r (1-F(x))^{n-r}, i=1, 2, 3, \dots, n$$

$$F_{(i:n)-ML-II-Exp}(x) = \sum_{r=i}^n \binom{n}{r} \left(1 - \left(\frac{e^{-\theta x}}{1-\alpha+ae^{-\theta x}} \right)^\beta \right)^r \left(\left(\frac{e^{-\theta x}}{1-\alpha+ae^{-\theta x}} \right)^\beta \right)^{n-r} \quad (32)$$

Furthermore, the *minimum* and *maximum* OS of X follows directly from (28) with $i=1$ and $i=n$, respectively.

Inference

In this section, we estimate the parameters of the ML-II-Exp distribution by following the method of MLE, as this method provides the maximum information about the unknown model parameter. Let $X_1, X_2, X_3, \dots, X_n$ be a random sample of size n from the ML-II-Exp distribution, then the likelihood function $L_{ML-II-Exp}(\psi = \alpha, \beta, \theta) = \prod_{i=1}^n f_{ML-II-Exp}(x_i; \psi)$ of X is given by

$$L_{ML-II-Exp}(\psi) = \prod_{i=1}^n \frac{(1-\alpha)\beta\theta e^{-\beta\theta x_i}}{(1-\alpha + \alpha e^{-\theta x_i})^{\beta+1}}$$

The log-likelihood function, $l(\psi)$ is given by

$$l_{ML-II-Exp}(\psi) = n(\log(1-\alpha) + \log\beta + \log\theta) - \beta\theta \sum_{i=1}^n x_i - (\beta+1) \sum_{i=1}^n \log(1-\alpha + \alpha e^{-\theta x_i}) \quad (33)$$

Partial derivatives of (33) w.r.t. α, θ , and β yield, respectively

$$\frac{\partial l_{ML-II-Exp}(\psi)}{\partial \alpha} = -\frac{n}{1-\alpha} - (\beta+1) \sum_{i=1}^n \left(\frac{-1 + e^{-\theta x_i}}{(1-\alpha + \alpha e^{-\theta x_i})} \right) = 0$$

$$\frac{\partial l_{ML-II-Exp}(\psi)}{\partial \beta} = \frac{n}{\beta} - \theta \sum_{i=1}^n x_i - \sum_{i=1}^n \log(1-\alpha + \alpha e^{-\theta x_i}) = 0$$

Table 3. Descriptive statistics.

Dataset	Minimum	1 st Quartile	Mean	3 rd Quartile	95% Confidence Interval	Maximum
1	0.040	1.866	2.563	3.376	(2.32,2.80)	4.663
2	0.046	1.122	2.085	2.820	(1.77,2.40)	5.140
3	0.390	1.840	2.621	3.220	(2.42,2.82)	5.560
4	5.000	55.00	68.33	80.25	(63.89,72.77)	147.0

Table 4. Average MLEs and standard errors (in parenthesis).

n	S-V Estimates (Standard Errors)			S-VI Estimates (Standard Errors)			S-VII Estimates (Standard Errors)		
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
25	0.8895 (0.0979)	0.4935 (0.4787)	1.3017 (0.9226)	0.4036 (0.4076)	0.6612 (1.9725)	0.9195 (2.4506)	0.8031 (0.3244)	0.0764 (0.0725)	0.1573 (0.1428)
50	0.9303 (0.0397)	0.6702 (0.3945)	1.1816 (0.4331)	0.5499 (0.2228)	0.5939 (0.5831)	1.4548 (1.2127)	0.9785 (0.0399)	0.0854 (0.6502)	0.2136 (0.1452)
100	0.8997 (0.0408)	0.5616 (0.2859)	1.1073 (0.3819)	0.4599 (0.1847)	0.8215 (0.8319)	1.0985 (0.9456)	0.9696 (0.0535)	0.0597 (0.0481)	0.1929 (0.1445)
200	0.9261 (0.0236)	0.4642 (0.1344)	1.2698 (0.2561)	0.3174 (0.1657)	0.4879 (0.5244)	1.2621 (1.2587)	0.8903 (0.0803)	0.0573 (0.0239)	0.1927 (0.0761)
300	0.9199 (0.0235)	0.3574 (0.0871)	1.3011 (0.2391)	0.5149 (0.1087)	0.3991 (0.2764)	1.6572 (1.0288)	0.9198 (0.0476)	0.0700 (0.0221)	0.1755 (0.0516)
400	0.9098 (0.0224)	0.3528 (0.0772)	1.3579 (0.2252)	0.4738 (0.0886)	0.6453 (0.3588)	1.1241 (0.5378)	0.9415 (0.0391)	0.0552 (0.0183)	0.2169 (0.0678)
500	0.9281 (0.0145)	0.4973 (0.0996)	1.1945 (0.1629)	0.5620 (0.0659)	0.7147 (0.3690)	1.0924 (0.4628)	0.9606 (0.0240)	0.0547 (0.0139)	0.2175 (0.0521)

$$\frac{\partial l_{ML-II-Exp}(\psi)}{\partial \theta} = \frac{n}{\theta} - \beta \sum_{i=1}^n x_i - (\beta + 1) \sum_{i=1}^n \left(\frac{-\alpha \theta e^{-\theta x_i}}{(1 - \alpha + \alpha e^{-\theta x_i})} \right) = 0$$

The maximum likelihood estimates ($\hat{\psi}_i = \hat{\alpha}, \hat{\beta}, \hat{\theta}$) of the ML-II-Exp distribution can be obtained by maximizing (33) or by solving the above non-linear equations simultaneously. These non-linear equations, however, do not provide an analytical solution for the MLEs and the optimum value of α , β and θ . Consequently, iterative techniques such as the Newton-Raphson type algorithm are an appropriate choice in the support of MLEs.

Simulation study

In this sub-section, we discuss the performance of MLEs using the following algorithm.

Step 1: A random sample $x_1, x_2, x_3, \dots, x_n$ of sizes $n = 25, 50, 100, 200, 300, 400$, and 500 are generated from (26).

Step 2: The required results are obtained based on the different combinations of the model parameters placed in S-V ($\alpha = 0.9, \theta = 1.1, \beta = 0.5$), S-VI ($\alpha = 0.5, \theta = 1.2, \beta = 0.6$), and S-VII ($\alpha = 0.9, \theta = 0.2, \beta = 0.06$), S-VIII ($\alpha = 0.5, \theta = 0.09, \beta = 0.2$), S-IX ($\alpha = 0.1, \theta = 0.1, \beta = 0.2$) and S-X ($\alpha = 0.9, \theta = 0.1, \beta = 0.9$).

Table 5. Bias, Root Mean Square Error (RMSE), variance, and mean.

n	Est.	S-VIII			S-IX			S-X		
		α	β	θ	α	β	θ	α	β	θ
25	Bias	0.1963	1.5078	1.1519	0.4068	1.2949	1.0298	-0.0001	2.0924	1.4949
	RMSE	0.1879	29.3727	5.4729	0.3575	25.1695	5.7784	0.0282	42.1068	12.3474
	Var.	0.1494	27.0993	4.1459	0.1921	23.4925	4.7180	0.0282	37.7286	10.1124
	Mean	0.6926	1.5978	1.3519	0.5068	1.3949	1.2298	0.8998	2.1924	2.3949
50	Bias	0.1717	0.7480	1.0657	0.3391	0.8762	0.8289	0.0125	0.4615	1.2245
	RMSE	0.1431	11.0845	6.1472	0.2802	14.4779	5.1008	0.0118	6.3089	9.8381
	Var.	0.1135	10.5251	5.0115	0.1652	13.7103	4.4136	0.0116	6.0959	8.3388
	Mean	0.6717	0.8380	1.2657	0.4391	0.9761	1.0289	0.9125	0.5615	2.1245
100	Bias	0.1453	0.6293	0.9190	0.2736	0.6040	0.8120	0.0077	0.1359	0.5918
	RMSE	0.1038	8.1899	6.7048	0.2046	8.8975	7.0779	0.0070	1.5761	3.3296
	Var.	0.0827	7.7939	5.8601	0.1297	8.5325	6.4185	0.0069	1.5575	2.9795
	Mean	0.6453	0.7193	1.1191	0.3736	0.7041	1.0120	0.9077	0.2359	1.4918
200	Bias	0.0929	0.2690	0.4411	0.1972	0.3435	0.5571	0.0047	0.0075	0.2014
	RMSE	0.0673	2.1801	3.3810	0.1351	3.7753	6.3946	0.0045	0.0046	0.4451
	Var.	0.0587	2.1078	3.1864	0.0962	3.6573	6.0841	0.0044	0.0046	0.4046
	Mean	0.5929	0.3590	0.6411	0.2972	0.4435	0.7571	0.9047	0.1075	1.1014
300	Bias	0.0793	0.1599	0.2405	0.1558	0.2281	0.4264	0.0049	0.0027	0.1191
	RMSE	0.0490	0.9116	0.8901	0.0993	1.5787	5.4937	0.0028	0.0018	0.1716
	Var.	0.0427	0.8861	0.8322	0.0751	1.5267	5.3118	0.0028	0.0018	0.1574
	Mean	0.5794	0.2499	0.4405	0.2558	0.3281	0.6264	0.9049	0.1028	1.0191
400	Bias	0.0606	0.1091	0.1665	0.1354	0.2417	0.3892	0.0027	0.0021	0.0736
	RMSE	0.0363	0.4973	0.4553	0.0843	1.3273	4.3197	0.0021	0.0012	0.0872
	Var.	0.0327	0.4854	0.4276	0.0660	1.2688	4.1683	0.0021	0.0012	0.0818
	Mean	0.5606	0.1991	0.3665	0.2354	0.3417	0.5892	0.9027	0.1021	0.9736
500	Bias	0.0506	0.0529	0.1483	0.1131	0.1231	0.3047	0.0029	0.0006	0.0611
	RMSE	0.0295	0.0968	0.4917	0.0642	0.3419	3.8841	0.0016	0.0007	0.0624
	Var.	0.0269	0.0968	0.4697	0.0514	0.3268	3.7912	0.0015	0.0007	0.0586
	Mean	0.5506	0.1429	0.3482	0.2131	0.2231	0.5047	0.9029	0.1007	0.9612

Step 3: Average MLEs and their corresponding standard errors (SEs) (in parenthesis) are presented in Table 4.

Step 4: Estimated bias, root mean square error (RMSE), variance, and mean values are presented in Table 5.

Step 5: Each sample is replicated $N = 500$ times.

Step 6: A gradual decrease in SEs, biases, RMSE, variances, means, and MLEs pretty close to the true parameters are observed with increase in the sample sizes.

Step 7: Finally, the estimates present in Tables 4 and 5 help us to specify that the method of maximum likelihood works consistently for the ML-II-Exp distribution.

The in-practice measures for the development of average estimate (AE), SE, bias, and RMSE are given as follows:

$AE(\hat{\psi}) = \frac{1}{N} \sum_{i=1}^N \hat{\psi}_i,$	$SE(\hat{\psi}) = \sqrt{\frac{1}{N} \sum_{i=1}^N (\hat{\psi}_i - \bar{\psi})^2},$	$RMSE(\hat{\psi}) = \sqrt{\frac{1}{N} \sum_{i=1}^N (\hat{\psi}_i - \psi)^2},$
$Bias(\hat{\psi}) = \frac{1}{N} \sum_{i=1}^N (\hat{\psi}_i - \psi),$	and $CI(\hat{\psi}) = \hat{\psi} \pm Z_{\frac{\alpha}{2}} \sqrt{var(\hat{\psi})}$	

Four real data applications

In this section, we explore four suitable lifetime datasets to model the ML-II-Exp distribution. These datasets are associated with the engineering sector. The first dataset relates to the study of failure times of 84 windshields for a particular model of aircraft (the unit for measurement is 1000 hours) that was first discussed by Ramos *et al.*¹⁸ The second dataset relates to the study of service times of 63 aircraft windshields (the unit for measurement is 1000 hours) that was discussed by Tahir *et al.*¹⁹ The third dataset follows the discussion of the breaking stress of carbon fibers (in Gba) that was initially developed by Nicholas and Padgett²⁰ and finally the fourth one relates to the study of fatigue life of 6061 - T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 101 observations with maximum stress per cycle of 31,000 psi. This dataset was pioneered by Birnbaum and Saunders.²¹ Subsequently, this data was discussed by Shanker *et al.*,²² after subtracting 65 from each observation. The datasets are given in the underlying data.

Table 6. List of CDFs for some competitive models.

Abbr.	Model	Parameter/ variable range $0 < x < \infty$	Reference
HL-Exp	$G_I(x) = \frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}$	$\alpha > 0$	Balakrishnan ²³
E-Exp	$G_{II}(x) = (1 - e^{-\alpha x})^\beta$	$\alpha, \beta > 0$	Ahuja and Nash ²⁴
Exp	$G_{III}(x) = 1 - e^{-\alpha x}$	$\alpha > 0$	Nadarajah and Kotz ²⁵
ETr-Exp	$G_{IV}(x) = 1 - e^{-\alpha(1-e^{-\beta})x}$	$\alpha, \beta > 0$	El-Alosey ²⁶
Logis-Exp	$G_V(x) = 1 - \frac{1}{1+(e^{\alpha x}-1)^\beta}$	$\alpha, \beta > 0$	Lan and Leemis ²⁷
MO-Exp	$G_{VI}(x) = 1 - \frac{ae^{-x}}{1-(1-\alpha)e^{-x}}$	$\alpha > 0$	Salah <i>et al.</i> ²⁸
NH-Exp	$G_{VII}(x) = 1 - e^{1-(1+\alpha x)^\beta}$	$\alpha, \beta > 0$	Nadarajah and Haghighi ²⁹
Ts-Exp	$G_{VIII}(x) = (1+\lambda)(1 - e^{-\alpha x}) - \lambda(1 - e^{-\alpha x})^2$	$\alpha > 0, \lambda \leq 1$	Merovci and Puka ³⁰
DUS-Exp-Exp	$G_{IX}(x) = \frac{e^{(1-e^{-\alpha x})}-1}{e-1}$	$\alpha > 0$	Kumar <i>et al.</i> ¹¹
Alp-Exp	$G_X(x) = \frac{\alpha(1-e^{-\beta x})-1}{\alpha-1}$	$\alpha, \beta > 0$	Mahdavi and Kundu ¹²

Half Logistic Exponential = HL-Exp, Exponentiated Exponential = E-Exp, Exponential = Exp, Erlang Truncated Exponential = ETr-Exp, Logistic-Exponential = Logis-Exp, Nadarajah-Haghighi Exponential = NH-Exp, Transmuted Exponential = Ts-Exp, Marshall-Olkin Exponential = MO-Exp, DUS Exponential Exponential = DUS-Exp-Exp, Alpha Power Exponential = Alp-Exp.

Table 7. Parameter estimates and standard errors (in parenthesis) along with goodness-of-fit for the failure time of 84 windshield dataset.

Model	Parameters (Standard Errors)				Goodness-of-fit			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	-LL	W*	A*	K-S
ML-II-Exp	0.8344 (0.1142)	0.9941 (0.0025)	15.3305 (9.570)	-	128.7357	0.1353	0.8537	0.0888
Logis-Exp	0.3946 (0.0193)	3.9081 (0.3917)	-	-	131.3591	0.1064	0.7073	0.0862
MO-Exp	11.2200 (1.9813)	-	-	-	134.2496	0.0479	0.4734	0.0988
Alp-Exp	91.0358 (45.7226)	0.8149 (0.0607)	-	-	135.6613	0.0693	0.6914	0.1191
E-Exp	0.7593 (0.0764)	3.5949 (0.6139)	-	-	141.3958	0.2187	1.7391	0.1215
NH-Exp	0.0062 (0.0073)	44.9430 (52.8504)	-	-	145.4100	0.0613	0.6058	0.2588
Ts-Exp	0.5717 (0.0470)	-	-	-0.9970 (0.0503)	146.8122	0.1818	1.5000	0.1855
HL-Exp	0.5776 (0.0497)	-	-	-	153.6200	0.1000	0.9368	0.2632
Exp-Exp	0.5137 (0.0469)	-	-	-	156.1826	0.1169	1.0577	0.2699
Exp	2.5609 (0.2776)	-	-	-	164.9877	0.1664	1.3972	0.3033
ETr-Exp	0.5431 (-)	1.2678 (-)	-	-	164.9877	0.1664	1.3972	0.3033

Table 8. Parameter estimates and standard errors (in parenthesis) along with goodness-of-fit for the service times of 63 aircraft windshield dataset.

Model	Parameters (Standard Errors)				Goodness-of-fit			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	-LL	W*	A*	K-S
ML-II-Exp	0.7013 (0.4333)	0.9260 (0.0884)	3.4208 (6.852)	-	98.1497	0.0350	0.2350	0.0657
MO-Exp	5.9950 (1.2886)	-	-	-	99.1004	0.0553	0.3428	0.0784
Alp-Exp	28.4623 (22.5320)	0.8771 (0.1036)	-	-	100.3555	0.0975	0.5932	0.1060
NH-Exp	0.0118 (0.0098)	26.4046 (21.4902)	-	-	100.1846	0.0638	0.3902	0.1436
Ts-Exp	0.6790 (0.0734)	-	-	-0.8693 (0.1465)	102.9673	0.1679	1.0249	0.1426
E-Exp	0.6921 (0.0942)	1.8979 (0.3402)	-	-	103.5466	0.2034	1.2315	0.1438
Logis-Exp	0.4922 (0.0379)	2.8590 (0.3664)	-	-	103.5850	0.0488	0.4020	0.0804
HL-Exp	0.6871 (0.0701)	-	-	-	103.8485	0.1201	0.7302	0.1643
Exp-Exp	0.6178 (0.0665)	-	-	-	105.0672	0.1393	0.8460	0.1699
ETr-Exp	0.6989 (7.1557)	1.1588 (22.3832)	-	-	109.2985	0.1861	1.1264	0.2077
Exp	2.0848 (0.2626)	-	-	-	109.2985	0.1861	1.1264	0.2077

Table 9. Parameter estimates and standard errors (in parenthesis) along with goodness-of-fit for the breaking stress of carbon fibers dataset.

Model	Parameters (Standard Errors)				Goodness-of-fit			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	-LL	W*	A*	K-S
ML-II-Exp	1.6982 (0.3168)	0.9869 (0.0056)	0.9675 (0.4008)	-	142.1201	0.0651	0.4142	0.0602
Logis-Exp	0.3850 (0.0148)	4.5316 (0.4141)	-	-	143.3108	0.0668	0.4723	0.0575
E-Exp	1.0133 (0.0875)	7.7926 (1.4973)	-	-	146.1823	0.2267	1.1861	0.1077
Ts-Exp	0.7651 (0.0704)	-	-	-2.0285 (0.1786)	150.0987	-	-	0.1841
MO-Exp	12.0519 (1.9083)	-	-	-	153.6640	0.0647	0.3791	0.1450
Alp-Exp	106.8843 (41.7742)	0.8280 (0.0528)	-	-	153.8885	0.1299	0.6566	0.1431
NH-Exp	0.0058 (0.0034)	48.1147 (28.6443)	-	-	171.4835	0.0691	0.4253	0.2896
H-Log-Exp	0.5709 (0.0451)	-	-	-	181.5941	0.1147	0.5874	0.2904
Exp-Exp	0.5061 (0.0425)	-	-	-	184.8960	0.1271	0.6482	0.2949
Exp	2.6210 (0.2621)	-	-	-	196.3709	0.1493	0.7643	0.3206
ETr-Exp	0.5467 (-)	1.1969 (-)	-	-	196.3709	0.1493	0.7643	0.3206

Table 10. Parameter estimates and standard errors (in parenthesis) along with goodness-of-fit for the fatigue life of 6061 - T6 aluminum coupons dataset.

Model	Parameters (Standard Errors)				Goodness-of-fit			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	-LL	W*	A*	K-S
ML-II-Exp	0.0786 (0.0129)	0.9954 (0.0015)	1.0426 (0.4383)	-	450.4221	0.0313	0.2384	0.0446
Logis-Exp	0.0147 (0.0004)	5.5022 (0.4889)	-	-	450.7880	0.0323	0.2447	0.0442
E-Exp	0.0413 (0.0033)	9.6723 (1.8416)	-	-	463.8919	0.2790	1.6812	0.1162
Ts-Exp	0.0232 (0.0015)	-	-	-1.2479 (0.0444)	488.3510	-	-	0.2308
MO-Exp	24668.86 (-)	-	-	-	5831.908	4.8685	22.758	0.9899
Alp-Exp	119.9019 (44.8694)	0.0316 (0.0019)	-	-	473.8240	0.1063	0.6380	0.1863
NH-Exp	0.0025 (0.0003)	4.7152 (0.6756)	-	-	500.5346	0.0910	0.6214	0.3567
H-Log-Exp	0.0221 (0.0017)	-	-	-	506.3137	0.1165	0.7363	0.3365
Exp-Exp	0.0195 (0.0016)	-	-	-	509.9945	0.1308	0.8179	0.3407
Exp	68.31 (6.8289)	-	-	-	522.4349	0.1646	1.0197	0.3667
ETr-Exp	0.02015 (0.0209)	1.2924 (2.7465)	-	-	522.4349	0.1646	1.0199	0.3667

The ML-II-Exp distribution is compared with the well-known models (CDF list is mentioned in Table 6). We follow some recognized selection criteria including -Log-likelihood (-LL), Anderson-Darling (A*), Cramer-Von Mises (W*), and Kolmogorov-Smirnov (K-S) test statistics. Some common results of descriptive statistics such as minimum value, 1st quartile, means, 3rd quartile, 95% confidence interval, and the maximum value, are tabulated in Table 3. The parameter estimates, standard errors (in parenthesis), and the goodness-of-fit are confirmed in Tables 7-10, respectively. The minimum value of the goodness-of-fit is the criteria of the better fit model that the ML-II-Exp distribution perfectly satisfies. Hence, we affirm that the ML-II-Exp distribution is a better fit than its competitors.

Furthermore, for a visual comparison the fitted density and distribution functions, Kaplan-Meier survival and probability-probability (PP) plots, total time on test transform (TTT), and box plots, are presented in Figures 3–6 (a, b, c, d, e, f, g, h), respectively. These plots provide sufficient information about the closest fit to the data. All the numerical results in the subsequent tables are calculated with the assistance of statistical software RStudio-1.2.5033. with its package AdequacyModel. The explored datasets are given in the underlying data.

Figure 3. Failure time of 84 windshields dataset.

Figure 4. Service time of 63 aircraft windshield dataset.

Figure 5. Breaking stress of carbon fibers dataset.

Figure 6. Fatigue life of 6061 - T6 aluminum coupons dataset.

Conclusion

In this article, we introduced and studied a more flexible G class, called the modified Lehmann Type-II (ML-II) G class of distributions along with explicit expressions for the moments, quantile function, and OS. The exponential distribution was used as the baseline distribution for ML-II-G class, known as ML-II-Exp distribution. It was discussed comprehensively, which demonstrated the reversed-J, constant, unimodal, and right-skewed shapes of a density function. The method of MLE along with the simulation was carried out to investigate the performance of the proposed method. The efficiency of the ML-II-G class was evaluated when the most efficient and consistent results of ML-II-Exp distribution competed the well-known models and explored the dominance along with a better fit in four real-life datasets.

We hope that in the future, the proposed class and its sub-models will explore the wider range of applications in diverse areas of applied research and will be considered as a choice against the baseline models.

Data availability

Figshare. four_dataset.csv. DOI: <https://doi.org/10.6084/m9.figshare.14518383.v1>³¹

This project contains the following extended data:

- Failure Time of 84 Windshield, Service Times of 63 Aircraft Windshield, Breaking Stress of Carbon Fibers and Fatigue Life of 6061 - T6 Aluminum Coupons used for the article titled “A New Modified Lehmann Type – II G Class of Distributions: Exponential Distribution with Theory, Simulation, and Applications to Engineering Sector”
- Ramos et al. dataset
- Tahir et al. dataset
- Nicholas and Padgett dataset
- Birnbaum and Saunders dataset

Data are available under the terms of the [Creative Commons Zero “No rights reserved” data waiver](#) (CC BY 4.0 Public domain dedication).

Acknowledgments

The authors are grateful to the Editor-in-Chief and anonymous referrers for their constructive comments and valuable suggestions which certainly improved the quality of the paper. We dedicate this work to our mentor Professor Dr. Munir Ahmad (Late) who was the Founding President of the Islamic Society of Statistical Sciences (ISOSS), Editor-in-Chief of Pakistan Journal Statistics, and the paramedic staff that are scarifying their lives against the COVID-19 pandemic war.

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Nofiu Idowu Badmus

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My comments on the manuscript are as follows:

Generally, the paper is on non-normal data (i.e skewed data) as one of the challenges in Statistics in which several authors in literature have developed many suitable distributions/models to capture the excessive skewness and kurtosis of the data. Although, the authors of this paper based their work on a part/segment in parametric distribution using G-model as the basic. They discussed and explored numerous G-models in literature to support their research. Properties of the ML-II G model are presented. The model is tested using both simulation and real-life phenomena. The results show that their model supersedes other models considered.

- The title should be **"A new modified Lehmann type - II G class of distributions: Theory, Simulation, and Applications"**.
- The abstract contains the problem, methods used, findings, and result which is good and ok.
- In the introduction, the CDFs of cited papers are not necessary to appear in the context except the CDF and PDF of the work they followed in their work. Therefore, the introduction should dwell more on the paper(s) they used. This implies that there is a need for them to upgrade/add more to the introduction.
- A new G class is developed which is known as a new modified Lehmann Type-II-(ML-II) G class of distributions. Why a new G class? Since we can see many G classes in the literature. What makes your G class better than others?
- Under Order Statistics, after the second expression where you wrote *"one may determine it by integrating (5-6)" - with respect to?* This is missing.
- Under limiting behavior, it should be: *"Here, we obtain the limiting behavior of PDF, CDF..."*.

- In Table 2, it should be that some chosen parameters with several values are presented.
- In the paper, it was discussed that lessing/setting one or two parameter(s) in the model led to some special cases emanating from the proposed model, One of those special cases is Lehmann Type II. Due to the fact that the author's work targeted Lehmann Type II they need to include/cite some Lehmann Type II research works from the literature review. The following articles can serve as a guide for authors:
 - **N. I. Badmus**, T. A. Bamiduro and S. G. Ogunobi (2014). Lehmann Type II Weighted Weibull Distribution¹.
 - **Badmus, N. Idowu** and Bamiduro, T. Adebayo. (2015). Log-Lehmann Type II Weighted Weibull (LLWW) Regression Model: Theory and Method².
 - **Badmus, N. I.**, Bamiduro, T. A., Akingbade, A. A and Rauf-Animasaun, M. A (2016). Estimation of Parameters in Generalized Modified Weighted Weibull Distribution³.
 - **Badmus, N. I.**, Alakija, T. O. and Olanrewaju, G. O. (2017). The Beta-Modified Weighted Rayleigh Distribution: Application to Virulent Tubercle Disease⁴.
- Authors may source for other ones online.

The manuscript can be accepted subject to the corrections listed above. Thanks.

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Is the work clearly and accurately presented and does it cite the current literature?

Yes

Is the study design appropriate and is the work technically sound?

Yes

Are sufficient details of methods and analysis provided to allow replication by others?

Yes

If applicable, is the statistical analysis and its interpretation appropriate?

Yes

Are all the source data underlying the results available to ensure full reproducibility?

Partly

Are the conclusions drawn adequately supported by the results?

Yes

Competing Interests: No competing interests were disclosed.

Reviewer Expertise: Inferential statistics, Computational Statistics, Distribution Theory

I confirm that I have read this submission and believe that I have an appropriate level of expertise to confirm that it is of an acceptable scientific standard, however I have significant reservations, as outlined above.

Reviewer Report 22 December 2021

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Frank Gomes-Silva

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The authors propose a new probabilistic class. Then, starting from this class, they propose a new distribution called ML-II-Exp. Present mathematical properties, simulation, and application for this new distribution. Here are some notes:

- The introduction can be improved. More motivation is needed to introduce a new generator. More references should be added, for example:
 - *Parameter induction in continuous univariate distributions: Well established G families*, by Tahir and Nadarajah¹; *Compounding of distributions: a survey and new generalized classes*, by Tahir and Cordeiro², among others. These two references are important in giving readers an overview of the area.
- The authors should have explored Lehmann's distributions - see, for example, *My Musings on a Pioneering Work by Erich Lehmann and Its Rediscoveries on Some Families of Distributions*, by Narayanaswamy Balakrishnan³. Professor Balakrishnan gives an explanation of Lehmann's contribution to the theory of distributions.
- Another interesting article: *Method for generating distributions and classes of probability distributions: the univariate case*⁴, where authors generalize (and introduce new generators) to the vast majority of classes; class T-X is a particular case of this new method. It is worth noting that for the first time new generators are proposed from multiple baselines.

- What do you mean by DUS (in the introduction)?
- On page 3 (just after Eq. 1), appears $\alpha < 1$ and then $\alpha > 0$. What does that mean?
- Identifiable problems are inconvenient when estimating the parameters of a model. A section to present proof of the model's (or even class') identifiability would be of interest. The following article presents an interesting section for identifiable issues: *Normal-G Class of Probability Distributions: Properties and Applications*⁵.
- Equation (22) appears in an unconventional form. The same goes for writing Equations (23) and (28). Writing in a conventional way!
- At the end of the equations, put a period. Equations are part of the text.
- In the simulation section, some discussions appear as steps. This can be confused with steps in an algorithm (which is usually placed in this type of section). The authors have presented an algorithm explaining a simulation. For example, what was done with the simulations that didn't converge?
- The simulation can be improved. Note that the mean (for the θ parameter) is not close to the true value of the parameter.
- Standard errors play an important role in terms of accuracy. Standard errors are high in the first application for Alp-Exp and NH-Exp models. Where are the standard errors for the Etr-Exp model (in the first application)?
- The W^* and A^* statistics are important measures of goodness-of-fit in distribution theory. But the proposed model (in the first application) loses to several others based on these indicators.
- In Table 8, the standard error of the Etr-Exp model is too high.
- In Table 9, the estimate for the parameter λ is inconsistent since by definition λ must be between -1 and 1.
- In Table 9, where are the standard errors for the Etr-Exp model?
- Table 10 gives an estimate (for λ) of -1.2479. This estimate is inconsistent since λ must be between -1 and 1. For this same model (in this same line), the W^* and A^* statistics do not appear. Again, a very high standard error appears for the Etr-Exp model.

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Is the work clearly and accurately presented and does it cite the current literature?

Partly

Is the study design appropriate and is the work technically sound?

Partly

Are sufficient details of methods and analysis provided to allow replication by others?

Partly

If applicable, is the statistical analysis and its interpretation appropriate?

Partly

Are all the source data underlying the results available to ensure full reproducibility?

Yes

Are the conclusions drawn adequately supported by the results?

Partly

Competing Interests: No competing interests were disclosed.

I confirm that I have read this submission and believe that I have an appropriate level of expertise to state that I do not consider it to be of an acceptable scientific standard, for reasons outlined above.

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