RESEARCH ARTICLE

A better approach to discuss medical science and engineering data with a modified Lehmann Type – II model [version 1; peer review: 1 approved with reservations]

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Abstract

Background: Modeling with the complex random phenomena that are frequently observed in reliability engineering, hydrology, ecology, medical science, and agricultural sciences was once thought to be an enigma. Scientists and practitioners agree that an appropriate but simple model is the best choice for this investigation. To address these issues, scientists have previously discussed a variety of bounded and unbounded, simple to complex lifetime models.

Methods: We discussed a modified Lehmann type II (ML-II) model as a better approach to modeling bathtub-shaped and asymmetric random phenomena. A number of complementary mathematical and reliability measures were developed and discussed. Furthermore, explicit expressions for the moments, quantile function, and order statistics were developed. Then, we discussed the various shapes of the density and reliability functions over various model parameter choices. The maximum likelihood estimation (MLE) method was used to estimate the unknown model parameters, and a simulation study was carried out to evaluate the MLEs' asymptotic behavior.

Results: We demonstrated ML- II's dominance over well-known competitors by modeling anxiety in women and electronic data.

Keywords

Power Function Distribution; Lehmann Type I, II Distribution; Failure Rate Function; Moments; Maximum Likelihood; Order Statistics; Quantile; Rényi Entropy.

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Any reports and responses or comments on the article can be found at the end of the article.

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Introduction

Over the last two decades, researchers' increasing interest in the development of new models has explored the remarkable characteristics of the baseline model. As a result, new models open new avenues for theoretical and applied researchers to address real-world problems, allowing them to fit asymmetric and complex random phenomena more proficiently and adequately. As a result, several modifications, extensions, and generalizations have been developed and discussed in the literature, with the Lehmann¹ type – I (L – I) and Lehmann type – II (L – II) models being among the simplest and most useful. The simple exponentiation of any arbitrary baseline model is given by L – I.

$$F(x; \alpha, \xi) = G^\alpha(x; \xi)$$

where $0 < x < 1$, and $\alpha > 0$ is a shape parameter.

Gupta *et al.*² are credited with the use of L–I on exponential distributions. On the other hand, Cordeiro *et al.*³ established the L–II–G class of distributions and developed a dual transformation of L–I, which is given by

$$F(x; \alpha, \xi) = 1 - (1 - G(x; \xi))^\alpha$$

where $G(x; \xi)$ is cumulative distribution function (CDF) of the arbitrary baseline model, based on the parametric vector ξ with $\alpha > 0$ as a shape parameter.

The closed-form feature of L–II allows one to derive and study its numerous properties, and in the literature, both approaches (L–I and L–II) have been extensively used in favor of the power function (PF) model, to study the unexplored characteristics of the classical and modified models. Recently, Arshad *et al.*^{4,5} developed bathtub-shaped failure rate and PF models followed by L–II and L–I families, respectively, and explored their applications in engineering data. Awodutire *et al.*⁶ generalized the half-logistic via L–II class, and Akilandeswari *et al.*⁷ proposed the Laplace L–I reliability growth model and discussed its application in the early detection of software failure based on time between failure observations.

The PF is a special case of the beta distribution in distribution theory, and its significance can be evaluated using statistical tests such as the likelihood ratio test. The PF's simplicity and utility has compelled researchers to investigate its further generalizations and applications in various fields of science. For this, we recommend that the reader look at Dallas's illustrious work.⁸ He discovered an intriguing relationship between PF and Pareto models when the inverse transformation of the Pareto variable explored the PF.⁸ Meanwhile, Meniconi and Barry⁹ discovered the PF as a best-fit model on electronic component data. Characterization is based on the independence of record values and order statistics, with lower record values attributed to Chang¹⁰ and Tavangar,¹¹ respectively. Cordeiro and Brito¹² created the beta version of the PF and discussed its use in petroleum reservoir and milk production data. The PF was characterized by Ahsanullah *et al.*¹³ using lower record values. Zaka *et al.*¹⁴ discussed techniques for estimating PF parameters such as least square (LS), relative least square (RLS), and ridge regression (RR). Tahir *et al.*¹⁵ generalized the PF via the Weibull-G class and applied it to bathtub-shaped data. Shahzad *et al.*¹⁶ used the techniques of L-, TL-, LL-, and LH moments to calculate the PF moments. Haq *et al.*¹⁷ generalized the PF via the QRTM-G class and investigated its application in two-lifetime data. Okorie *et al.*¹⁸ generalized the PF via the Marshall-Olkin-G class (Marshall and Olkin¹⁹) and investigated its application in data on anxiety in women and evaporation. Usman *et al.*²⁰ proposed an exponentiated version of transmuted PF and investigated its application in biological and engineering data. Hassan *et al.*²¹ generalized the PF by following the odd exponential-G class (Tahir *et al.*²²) and discussed its application in three-lifetime data. Zaka *et al.*²³ developed a new reflected PF and investigated its application in medical sciences data, while Al-Mutairi²⁴ discussed the weighted PF via the QRTM G-class and investigated its application in the engineering sector.

Modified Lehmann type II model

We developed a potentiated lifetime model known as the modified Lehmann type II (ML-II) model. It is constrained by the interval (0, 1). The ML-II is extremely well suited to modeling asymmetric and bathtub-shaped phenomena. By including a scale parameter ($\alpha > 0$), in the baseline model, it begins to outperform its competitors in terms of fit and robustness of the tail weight/skewness of the density function.

The ML-II is said to follow a random variable X if its associated CDF and corresponding probability density function (PDF) are given by

$$F_{\text{ML-II}}(x; \alpha, \beta) = 1 - \left(\frac{1-x}{1+\alpha x} \right)^\beta \quad (1)$$

and

$$f_{\text{ML-II}}(x; \alpha, \beta) = \frac{\beta(\alpha+1)(1-x)^{\beta-1}}{(1+\alpha x)^{\beta+1}} \quad (2)$$

where $0 < x < 1$, and $\alpha > -1, \beta > 0$ are the scale and shape parameters, respectively. For $\alpha = 0$, the ML-II reduces to the L-II (baseline model).

Balogun *et al.*,^{25,26} has developed the generalized version and G-class of ML-II (Equation (1)), respectively, and explored their applications in multidisciplinary areas of science.

We had the following objectives:

- (i) to develop a two-parameter model with an approach that had not been studied and discussed in the past;
- (ii) For the new model to have attractive closed-form features for CDF, PDF, and a likelihood function that is simple to interpret,;
- (iii) PDF and HRF to hold J, reversed-J, and bathtub shapes;
- (iv) To provide comparative results and a better fit than competing models.

This article is divided into the following sections: 'Linear representation' presents a mixture representation as well as numerous structural and reliability measures. 'Estimation' includes the estimation of model parameters using the maximum likelihood estimation (MLE) method, as well as a simulation study. 'Application' discusses real-world applications, and the final section summarizes the conclusion.

Linear representation

The linear representations of CDF and PDF make the calculations much easier than the traditional integral computation for determining the mathematical properties. We consider the binomial expansion for this.

$$(1-z)^\beta = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} z^i, |z| < 1$$

$$F_{\text{ML-II}}(x; \alpha, \beta) = 1 - (1-x)^\beta (1+\alpha x)^{-\beta}$$

$$F_{\text{ML-II}}(x; \alpha, \beta) = 1 - \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} x^i \sum_{j=0}^{\infty} \binom{-\beta}{j} (\alpha x)^j$$

Hence, the mixture representation of CDF is given as

$$F_{\text{ML-II}}(x; \alpha, \beta) = \sum_{i,j=0}^{\infty} \phi_{ij} x^{i+j}$$

PDF is given as follows

$$f_{\text{ML-II}}(x; \alpha, \beta) = \beta(1+\alpha) \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} x^i \sum_{j=0}^{\infty} \binom{\beta+1}{j} (\alpha x)^j$$

or

$$f_{\text{ML-II}}(x; \alpha, \beta) = \beta(1+\alpha) \sum_{i,j=0}^{\infty} \eta_{ij} x^{i+j}$$

where $\phi_{ij} = \binom{\beta}{i} \binom{-\beta}{j} (-1)^i \alpha^j, \alpha > -1, \beta > 0, \eta_{ij} = \alpha^j (-1)^i \binom{\beta-1}{i} \binom{\beta+1}{j}, \alpha > -1, \beta > 0$

Reliability characteristics

The probability model plays an important role in reliability engineering by analyzing and predicting the lifespan of a component. Notable contributions include the survival function $S(x)$, hazard rate function $h(x)$, cumulative hazard rate function $h_c(x)$, reverse hazard rate function $h_r(x)$, Mills ratio $M(x)$, and odd function $O(x)$.

The reliability function can be defined as the probability that a component will survive at time x . It is defined analytically as $S(x) = 1 - F(x)$. The survival/reliability function of X is denoted by

$$S_{\text{ML-II}}(x; \alpha, \beta) = \left(\frac{1-x}{1+\alpha x} \right)^\beta \quad (3)$$

Terms such as “failure rate function”, “hazard rate function”, and “force of mortality” are frequently discussed in the literature. These terms are used to describe the failure rate of a component over a specific time period (say x). It is mathematically defined as $h(x) = f(x)/R(x)$.

The failure rate function of X is denoted by

$$h_{\text{ML-II}}(x; \alpha, \beta) = \frac{\beta(\alpha+1)}{(1-x)(1+\alpha x)} \quad (4)$$

The mechanical components/parts of some systems are frequently assumed to follow the bathtub-shaped failure rate phenomenon. To discuss the significance of the MLII, several well-established and useful reliability measures are available in the literature. One of them is the cumulative hazard rate function, which is defined as $h_c(x) = -\log(S(x))$. The cumulative hazard rate function of X is denoted by

$$h_{c-\text{ML-II}}(x; \alpha, \beta) = -\log \left(\left(\frac{1-x}{1+\alpha x} \right)^\beta \right)$$

The reverse hazard rate function is defined by $h_r(x) = f(x)/F(x)$. The reverse hazard rate function of X is given by

$$h_{r-\text{ML-II}}(x; \alpha, \beta) = \frac{\beta(\alpha+1)(1-x)^{\beta-1}}{(1+\alpha x) \left((1+\alpha x)^\beta - (1-x)^\beta \right)}$$

Mills ratio is defined by $M(x) = S(x)/f(x)$. Mills ratio of X is given by

$$M_{\text{ML-II}}(x; \alpha, \beta) = \frac{(1+\alpha x)(1-x)^\beta}{\beta(\alpha+1)(1-x)^{\beta-1}}$$

The odd function is defined by $O(x) = F(x)/S(x)$. The odd function of X is given by

$$O_{\text{ML-II}}(x; \alpha, \beta) = \frac{(1+\alpha x)^\beta - (1-x)^\beta}{(1-x)^\beta}$$

As mentioned above, we can obtain the linear expression for reliability characteristics. In terms of linear expression, the reliability and failure rate functions of X are given by

$$S^*_{\text{ML-II}}(x; \alpha, \beta) = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} x^i \sum_{j=0}^{\infty} \binom{-\beta}{j} (\alpha x)^j$$

and

$$h^*_{\text{ML-II}}(x; \alpha, \beta) = \frac{\beta(1+\alpha) \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} x^i \sum_{j=0}^{\infty} \binom{\beta+1}{j} (\alpha x)^j}{\sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} x^i \sum_{j=0}^{\infty} \binom{-\beta}{j} (\alpha x)^j}$$

Limiting behavior

Propositions 1 and 2 discuss the limiting behavior of the ML-II's cumulative distribution (CDF), density (PDF), reliability ($S(x)$), and failure rate ($h(x)$) functions for $x \rightarrow 0$ and $x \rightarrow 1$.

Proposition-1. The limiting behavior of the CDF, PDF, $S(x)$, and $h(x)$ of the ML-II at $x \rightarrow 0$ is given below.

$$F_{\text{ML-II}}(0; \alpha, \beta) \sim 0$$

$$f_{\text{ML-II}}(0; \alpha, \beta) \sim \beta(1 + \alpha)$$

$$S_{\text{ML-II}}(0; \alpha, \beta) \sim 1$$

$$h_{\text{ML-II}}(0; \alpha, \beta) \sim \beta(1 + \alpha)$$

Proposition-2. The limiting behavior of CDF, PDF, $S(x)$, and $h(x)$ of the ML-II at $x \rightarrow 1$ is given as follows, respectively.

$$F_{\text{ML-II}}(1; \alpha, \beta) = 1$$

$$f_{\text{ML-II}}(1; \alpha, \beta) = 0$$

$$S_{\text{ML-II}}(1; \alpha, \beta) = 1$$

$$h_{\text{ML-II}}(1; \alpha, \beta) = 0$$

Shapes of density and failure rate functions

The possible shapes of the ML-II's density and failure rate functions are sketched over various model parameter choices shown in Figures 1 and 2. Figure 1 depicts the J, reverse-J, and bathtub shapes of the density function, while Figure 2 depicts the U, bathtub, and reverse-J shapes of the failure rate function.

Quantile, mode, skewness, and kurtosis

The concept of quantile function was introduced by Hyndman and Fan.²⁷ Inverting the CDF yields the p^{th} quantile function of ML-II (Equation (1)). The quantile function is defined as follows: $p = F(x_p) = P(X \leq x_p)$, $p \in (0, 1)$. Then, the quantile function of X is given by

$$x_{p-\text{ML-II}} = \frac{1 - (1 - p)^{1/\beta}}{1 + \alpha(1 - p)^{1/\beta}}, \alpha > -1, \beta > 0 \quad (5)$$

Put $p = 0.25, 0.50$, and 0.75 in Equation (3) to get the first quartile, median, and third quartile of X . To generate random numbers in the future, we will assume that the CDF in Equation (1) follows a uniform distribution $u = U. (0, 1)$.

Figure 1. Density function.

Figure 2. Failure rate function.

The mode of X is calculated by taking the first derivative of PDF (Equation (2)) and equating it to zero, as shown by

$$f'_{ML-II}(x; \alpha, \beta) = \frac{\beta(\alpha+1)(1-x)^\beta(1+\alpha x)^{-\beta-2}(2\alpha x - (1+\alpha)\beta - \alpha + 1)}{(1-x)^2} = 0$$

As a result, a simplified form of the mode is given by

$$\text{Mode}_{ML-II} = \frac{(\alpha+1)(\beta+1)-1}{2\alpha}, \alpha > -1, \beta > 0$$

Measures of skewness and kurtosis based on quartiles and octiles are less sensitive to outliers and work well against models, but they are deficient in moments.

<p>Bowley's²⁸ measure of skewness</p> $S_B = \frac{Q_{(\frac{3}{4})} + Q_{(\frac{1}{4})} - 2Q_{(\frac{1}{2})}}{Q_{(\frac{3}{4})} - Q_{(\frac{1}{4})}}$	<p>Moore's²⁹ measure of kurtosis</p> $K_M = \frac{Q_{(\frac{3}{8})} - Q_{(\frac{1}{8})} - Q_{(\frac{5}{8})} + Q_{(\frac{7}{8})}}{Q_{(\frac{3}{4})} - Q_{(\frac{1}{4})}}$
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According to Figures 3 and 4, the skewness and kurtosis plots of the ML-II may be positively skewed.

Moments and associated measures

Moments play a significant role in distribution theory, where they are used to discuss the various characteristics and important features of the probability model.

Figure 3. Skewness.

Figure 4. Kurtosis.

Theorem 1. Let $X \sim \text{ML-II}(x; \alpha, \beta)$, with $\alpha > -1, \beta > 0$, then the r -th ordinary moment (say μ'_r) of X is given by

$$\mu'_{r-\text{ML-II}} = E \sum_{i=0}^{\infty} D_i B(r+i+1, \beta)$$

where $E = \beta(1+\alpha)$, $D_i = \alpha^i \binom{-\beta-1}{i}$

Proof: r -th ordinary moment can be written by following Equation (2) as

$$\begin{aligned} \mu'_{r-\text{ML-II}} &= \beta(1+\alpha) \int_0^1 x^r (1-x)^{\beta-1} (1+\alpha x)^{-(\beta+1)} dx \\ \mu'_{r-\text{ML-II}} &= \alpha\beta\gamma(1+\alpha) \sum_{i=0}^{\infty} D_i \int_0^1 x^{r+i} (1-x)^{\beta-1} dx \end{aligned}$$

As a result, the above integral reduces to the r -th moment, which is given by

$$\mu'_{r-\text{ML-II}} = E \sum_{i=0}^{\infty} D_i B(r+i+1, \beta) \quad (6)$$

where $B(x; \alpha, \beta) = \int_0^x t^{\alpha-1} (1-t)^{\beta-1} dt$ is the beta function, $E = \beta(1+\alpha)$, $D_i = \alpha^i \binom{-\beta-1}{i}$ and $\alpha > -1, \beta > 0$

The r -th moment is quite helpful in the development of several statistics. For instance, the mean of X can be obtained by setting $r=1$ in Equation (6) and is given by

$$\mu'_{1-\text{ML-II}} = E \sum_{i=0}^{\infty} D_i B(2+i, \beta)$$

Moment generating function $M_X(t)$ is defined as $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$. It is obtained by following Equation (6) and is given by

$$M_{X-\text{ML-II}}(t) = E \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{i=0}^{\infty} D_i B(r+i+1, \beta)$$

Characteristic function is defined as $\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu'_r$. It is obtained by following Equation (6) and is given by

$$\phi_{X-\text{ML-II}}(t) = E \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{i=0}^{\infty} D_i B(r+i+1, \beta)$$

The factorial generating function of X is defined as $F_x(t) = E(1+t)^x = E(e^{x \ln(1+t)}) = \sum_{r=0}^{\infty} \frac{(\ln(1+t))^r}{r!} \mu'_r$. It is obtained by following Equation (6) and is given by

$$F_{x-ML-II}(t) = E \sum_{r=0}^{\infty} \frac{(\ln(1+t))^r}{r!} \sum_{i=0}^{\infty} D_i B(r+i+1, \beta)$$

The Fisher index $\frac{Var(X)}{E(X)}$ may play a supportive role in the discussion of variability in X and it is given by

$$\text{Fisher index}_{ML-II} = \frac{E \sum_{i=0}^{\infty} D_i B(3+i, \beta) - (E \sum_{i=0}^{\infty} D_i B(2+i, \beta))^2}{E \sum_{i=0}^{\infty} D_i B(2+i, \beta)}$$

For the negative moments of X , substitute r by $-w$ in Equation (6) and it is given by

$$\mu'_{-w-ML-II} = E \sum_{i=0}^{\infty} D_i B(-w+i+1, \beta)$$

Furthermore, for fractional positive and fractional negative moments of X , substitute r by $(\frac{m}{n})$ and $(-\frac{m}{n})$ in Equation (6), respectively.

The Mellin transformation is well-known in statistics as a product distribution as well as a quotient for independent random variables. The Mellin transformation is presented by $M_x(m) = E(x^{m-1}) = \int x^{m-1} f(x) dx$. The Mellin transformation of X is given by

$$M_{x-ML-II}(m) = E \sum_{i=0}^{\infty} D_i B(m+i, \beta).$$

where $B(x; \alpha, \beta) = \int_0^x t^{\alpha-1} (1-t)^{\beta-1} dt$ is the beta function, $E = \beta(1+\alpha)$, $D_i = \alpha^i \binom{-\beta-1}{i}$ and $\alpha > -1, \beta > 0$.

One may perhaps further determine the well-established statistics such as skewness ($\tau_1 = \mu'_3/\mu'_2$), and kurtosis ($\tau_2 = \mu'_4/\mu'_2$), of X by integrating Equation (6). A well-established relationship between the central moments (μ_s) and cumulants (K_s) of X may easily be defined by ordinary moments by $\mu_s = \sum_{k=0}^s \binom{s}{k} (-1)^k (\mu'_1)^s \mu'_{s-k}$. Hence, the first four cumulants can be calculated by $K_1 = \mu'_1$, $K_2 = \mu'_2 - \mu_1'^2$, $K_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$, and $K_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^4$.

Some numerical results of the first four ordinary moments ($\mu'_1, \mu'_2, \mu'_3, \mu'_4$), mode (a value that appears frequently in data), ε^2 = variance (a measure of dispersion), τ_1 = skewness (measure of asymmetry), and τ_2 = kurtosis (a measure to discuss the heaviness of the distribution tails) for some chosen parameters are presented in Table 1 for S-I ($\alpha = 0.5, \beta = 0.9$), S-II ($\alpha = 0.9, \beta = 1.5$), S-III ($\alpha = 1.2, \beta = 1.9$), S-IV ($\alpha = 3.2, \beta = 0.9$), and S-V ($\alpha = 0.2, \beta = 3.9$). Note that the results of

Table 1. Numerical results of moments (μ'_s), variance (ε^2), skewness (τ_1), and kurtosis (τ_2).

μ'_s	S-I	S-II	S-III	S-IV	S-V
μ'_1	0.460	0.295	0.224	0.302	0.181
μ'_2	0.297	0.143	0.089	0.163	0.056
μ'_3	0.221	0.086	0.046	0.110	0.023
μ'_4	0.176	0.058	0.028	0.067	0.011
Mode	1.850	2.083	2.241	1.091	12.200
ε^2	0.029	0.044	0.025	0.067	0.002
τ_1	0.010	0.100	0.492	0.028	1.292
τ_2	0.119	0.598	0.981	0.483	-0.076

moments and variance decrease gradually, while skewness falls between 0 and 1, mode increases, and kurtosis can be negative subject to the model parameter combinations.

Incomplete moments

Lower incomplete (LI) and upper incomplete moments are the two types of incomplete moments. LI moments are defined by $\Phi_r(t) = \int_0^t x^r f(x) dx$. The LI moments of X are given as

$$\Phi_{r-ML-II}(t) = E \sum_{i=0}^{\infty} D_i B_i(r+i+1, \beta) \quad (7)$$

The *residual life function* of random variable X, $R(t) = (X - t/X > t)$, is the likelihood that a component whose life says x, survives in the different time intervals at $t \geq 0$. Analytically, it can be written as follows:

$$S_{R(t)-ML-II}(x; \alpha, \beta) = \frac{S(x+t)}{S(t)}$$

Residual life function of X

$$S_{R(t)-ML-II}(x; \alpha, \beta) = \frac{(1 - (x+t))^\beta (1 + \alpha t)^\beta}{(1 + \alpha(x+t))^\beta (1 - t)^\beta}$$

with the associated CDF is given as follows

$$F_{R(t)-ML-II}(x; \alpha, \beta) = \frac{(1 + \alpha(x+t))^\beta (1 - t)^\beta - (1 - (x+t))^\beta (1 + \alpha t)^\beta}{(1 + \alpha(x+t))^\beta (1 - t)^\beta}$$

The *mean residual life* function of X is given by

$$E(S_{R-ML-II}(x; \alpha, \beta)) = \frac{1}{S(t)} \left(\mu'_1 - \int_0^t x f(x) dx \right) - t, t \geq 0$$

$$E(S_{R-ML-II}(x; \alpha, \beta)) = \frac{(1 + \alpha t)^\beta}{(1 - t)^\beta} \left(E \sum_{i=0}^{\infty} D_i (B(2+i, \beta) - B_i(2+i, \beta)) \right) - t$$

Further, the *reverse residual life* can be defined as $\bar{R}(t) = t - X/X \leq t$

$$S_{\bar{R}(t)-ML-II}(x; \alpha, \beta) = \frac{F(t-x)}{F(t)}, t \geq 0$$

Reverse residual life function of X

$$S_{\bar{R}(t)-ML-II}(x; \alpha, \beta) = \frac{(1 - (t-x))^\beta (1 + \alpha t)^\beta}{(1 + \alpha(t-x))^\beta (1 - t)^\beta}$$

with the associated CDF is given as follows

$$F_{\bar{R}(t)-ML-II}(x; \alpha, \beta) = \frac{(1 + \alpha(t-x))^\beta (1 - t)^\beta - (1 - (t-x))^\beta (1 + \alpha t)^\beta}{(1 + \alpha(t-x))^\beta (1 - t)^\beta}$$

The *mean reversed residual life* function/mean waiting time is given by

$$E(S_{\bar{R}-ML-II}(x; \alpha, \beta)) = t - \frac{1}{F(t)} \int_0^t x f(x) dx, t \geq 0$$

$$E(S_{\bar{R}-ML-II}(x; \alpha, \beta)) = t - \frac{(1 + \alpha x)^\beta}{(1 + \alpha x)^\beta - (1 - x)^\beta} E \sum_{i=0}^{\infty} D_i B_i(2 + i, \beta)$$

where $B(x; \alpha, \beta) = \int_0^x t^\alpha (1-t)^{\beta-1} dt$ is the beta function, $E = \beta(1 + \alpha)$, $D_i = \alpha^i \binom{-\beta-1}{i}$, and $\alpha > -1, \beta > 0$

Order statistics

Order statistics and its moments play an important role in reliability analysis and life testing of a component in quality control. Let $X_1, X_2, X_3, \dots, X_n$ be a random sample of size n following the ML-II model and $\{X_{(1)} < X_{(2)} < X_{(3)} < \dots < X_{(n)}\}$ be the corresponding order statistics. The random variables $X_{(i)}$, $X_{(1)}$, $X_{(n)}$ are the i -th, minimum, and maximum order statistics of X , respectively.

The PDF of the i -th order statistics is given by

$$f_{(i:n)}(x) = \frac{1}{B(i, n-i+1)!} (F(x))^{i-1} (1-F(x))^{n-i} f(x),$$

$$i = 1, 2, 3, \dots, n.$$

By incorporating Equations (1) and (2), the PDF of the i -th order statistics is given by

$$f_{(i:n)-ML-II}(x; \alpha, \beta) = \frac{\beta(\alpha+1)}{B(i, n-i+1)!} \left(1 - \left(\frac{1-x}{1+\alpha x}\right)^\beta\right)^{i-1} \left(\left(\frac{1-x}{1+\alpha x}\right)^\beta\right)^{n-i} \left(\frac{(1-x)^{\beta-1}}{(1+\alpha x)^{\beta+1}}\right) \quad (8)$$

Equation (8) can be written as

$$f_{(i:n)-ML-II}(x; \alpha, \beta) = \frac{1}{B(i, n-i+1)!} (\beta(\alpha+1)) \sum_{j=0}^{\infty} G_{ij} (-1)^i \alpha^j x^j (1-x)^{2\beta+i-1} \quad (9)$$

Straightforward computation of Equation (9) leads to the w -th moment order statistics of X and it is given by

$$E(X^w) = \mu'_{w-OS-ML-II} = \frac{1}{B(i, n-i+1)!} (\beta(\alpha+1)) \sum_{j=0}^{\infty} G_{ij} (-1)^i \alpha^j B(j+1, 2\beta+i) \quad (10)$$

where $G_{ij} = \binom{\beta}{i} \binom{-(\alpha\beta + \alpha + i)}{j}$, $\alpha > -1, \beta > 0$. Further, the minimum and maximum order statistics of X follow directly from Equation (8) with $i = 1$ and $i = n$, respectively.

Stress – strength reliability

Let X_1 and X_2 represent a component's strength and stress, respectively, following the same univariate distribution. The inadequacy or effectiveness of a component is dependent on whether $X_2 > X_1$ and $X_2 < X_1$, respectively. Stress – strength reliability can be written as $R = P(X_2 < X_1)$.

Theorem 2. Let $X_1 \sim \text{ML-II}(x; \alpha, \beta_1)$ and $X_2 \sim \text{ML-II}(x; \alpha, \beta_2)$ be independent ML-II distributed random variables; then the reliability R is defined as $\frac{\beta_2}{\beta_1 + \beta_2}$

Proof: Reliability R is defined as

$$R = \int f_1(x) F_2(x) dx$$

Reliability of X is given by

$$R_{ML-II} = \int \left(\frac{\beta_1(\alpha+1)(1-x)^{\beta_1-1}}{(1+\alpha x)^{\beta_1+1}} \right) \left(1 - \left(\frac{1-x}{1+\alpha x} \right)^{\beta_2} \right) dx.$$

Hence the above integral reduces R in terms of β_1 and β_2 , refers to the stress-strength reliability of the ML-II, and it is given by

$$R_{\text{ML-II}} = \frac{\beta_2}{\beta_1 + \beta_2} \cdot \beta_1, \beta_2 > 0$$

Entropy

There are a number of schools of thought about defining entropy measures. Entropy can be the quantity of disorder, randomness, diversity, or sometimes an uncertainty in a system.

The Rényi³⁰ entropy of X is defined by

$$I_\delta(X) = \frac{1}{1-\delta} \log \int_0^1 f^\delta(x) dx, \quad \delta > 0 \text{ and } \delta \neq 1$$

First, we simplify $f(x)$ in terms of $f^\delta(x)$ by considering Equation (2)

$$f_{\text{ML-II}}^\delta(X; \alpha, \beta, \delta) = \beta^\delta (\alpha + 1)^\delta (1 - X)^{\delta(\beta-1)} (1 + \alpha X)^{-\delta(\beta+1)}$$

by applying the binomial expansion to this equation, we get

$$f_{\text{ML-II}}^\delta(X; \alpha, \beta, \delta) = \beta^\delta (\alpha + 1)^\delta \sum_{i=0}^{\infty} \binom{-\delta(\beta+1)}{i} \alpha^i X^i (1 - X)^{\delta(\beta-1)}$$

and by placing this information in $I_\delta(X)$, we get

$$I_{\delta-\text{ML-II}}(X; \alpha, \beta, \delta) = \frac{1}{1-\delta} \log \left(\beta^\delta (\alpha + 1)^\delta \int_0^1 \sum_{i=0}^{\infty} \binom{-\delta(\beta+1)}{i} \alpha^i x^i (1-x)^{\delta(\beta-1)} dx \right)$$

hence, by integrating the last equation we obtain the reduced form of the Rényi entropy of X and it is given by

$$I_{\delta-\text{ML-II}}(X; \alpha, \beta, \delta) = \frac{1}{1-\delta} \log \left(\beta^\delta (\alpha + 1)^\delta \sum_{i=0}^{\infty} A_i B(\delta(\beta-1) + 1, i + 1) \right) \quad (11)$$

where $A_i = \binom{-\delta(\beta+1)}{i} \alpha^i, \alpha > -1, \beta > 0$.

A generalization of the Boltzmann-Gibbs entropy is the η – entropy, although in physics, it is referred to as the Tsallis entropy. The Tsallis³¹ entropy/ η – entropy is described by

$$H_\eta(X) = \frac{1}{\eta-1} \left(1 - \int_0^1 f^\eta(x) dx \right), \quad \eta > 0 \text{ and } \eta \neq 1.$$

The Tsallis Entropy of X is given by

$$H_{\eta-\text{ML-II}}(X; \alpha, \beta, \eta) = \frac{1}{\eta-1} \left(1 - \beta^\eta (\alpha + 1)^\eta \sum_{i=0}^{\infty} C_i B(\eta(\beta-1) + 1, i + 1) \right)$$

where $C_i = \binom{-\eta(\beta+1)}{i} \alpha^i, \alpha > -1, \beta > 0$.

Havrda and Charvat³² introduced ω – entropy measure. It is presented by

$$H_\omega(X) = \frac{1}{2^{\omega-1} - 1} \left(\int_0^1 f^\omega(x) dx - 1 \right), \quad \omega > 0 \text{ and } \omega \neq 1.$$

The Havrda and Charvat entropy of X is given by

$$H_{\omega-ML-II}(X; \alpha, \beta, \omega) = \frac{1}{2^{\omega-1} - 1} \left(\left(\beta^{\omega} (\alpha + 1)^{\omega} \sum_{i=0}^{\infty} F_i B(\omega(\beta - 1) + 1, i + 1) \right) - 1 \right)$$

where $F_i = \binom{-\omega(\beta+1)}{i} \alpha^i, \alpha > -1, \beta > 0$.

Arimoto³³ generalized the work of³² by introducing ε – entropy measure. It is presented by

$$H_{\varepsilon}(X) = \frac{1}{2^{\varepsilon-1} - 1} \left(\left(\int_0^1 f^{\frac{1}{\varepsilon}}(x) dx \right)^{\varepsilon} - 1 \right), \varepsilon > 0 \text{ and } \varepsilon \neq 1.$$

The Arimoto entropy of X is given by

$$H_{\varepsilon-ML-II}(X; \alpha, \beta, \varepsilon) = \frac{1}{2^{\varepsilon-1} - 1} \left(\left(\left(\beta^{\frac{1}{\varepsilon}} (\alpha + 1)^{\frac{1}{\varepsilon}} \times \sum_{i=0}^{\infty} G_i B\left(\frac{1}{\varepsilon}(\beta - 1) + 1, i + 1\right) \right)^{\varepsilon} - 1 \right) \right)$$

where $G_i = \binom{-\frac{1}{\varepsilon}(\beta+1)}{i} \alpha^i, \alpha > -1, \beta > 0$.

Boekee and Lubba³⁴ developed the τ – entropy measure. It is presented by

$$H_{\tau}(X) = \frac{\tau}{\tau - 1} \left(1 - \left(\int_0^1 f^{\tau-1}(x) dx \right)^{\frac{1}{\tau}} \right), \tau > 0 \text{ and } \tau \neq 1.$$

Boekee and Lubba entropy of X is given by

$$H_{\tau-ML-II}(X; \alpha, \beta, \tau) = \frac{\tau}{\tau - 1} \left(1 - \left(\beta^{(\tau-1)} (\alpha + 1)^{(\tau-1)} \times \sum_{i=0}^{\infty} I_i B((\tau - 1)(\beta - 1) + 1, i + 1) \right)^{\frac{1}{\tau}} \right)$$

where $I_i = \binom{-(\tau-1)(\beta+1)}{i} \alpha^i, \alpha > -1, \beta > 0$.

Mathai and Haubold³⁵ generalized the classical Shannon entropy known as ζ – entropy. It is presented by

$$H_{\zeta}(X) = \frac{1}{\zeta - 1} \left(\int_0^1 f^{2-\zeta}(x) dx - 1 \right), \zeta > 0 \text{ and } \zeta \neq 1.$$

The Mathai and Haubold entropy of X is given by

$$H_{\zeta-ML-II}(X; \alpha, \beta, \zeta) = \frac{1}{\zeta - 1} \left(\left(1 - \beta^{(2-\zeta)} (\alpha + 1)^{(2-\zeta)} \sum_{i=0}^{\infty} J_i B((2 - \zeta)(\beta - 1) + 1, i + 1) \right) - 1 \right)$$

where $J_i = \binom{-(2-\zeta)(\beta+1)}{i} \alpha^i, \alpha > -1, \beta > 0$.

Table 2 presents the flexible behavior of the entropy measures for some chosen model parameters for S-VI ($\alpha = 1.1, \beta = 2.1$), S-VII ($\alpha = 1.1, \beta = 1.5$), and S-VIII ($\alpha = 2.1, \beta = 3.5$).

Table 2. Numerical results of Rényi, Tsallis, Havrda and Charvat, Arimoto, Boekee and Lubba, and Mathai and Haubold entropy measures.

Entropy	Initial value	S-VI	S-VII	S-VIII	Status
Rényi	$\delta = 1.1$	9.977	4.330	7.023	Decreasing
	$\delta = 1.5$	2.721	1.181	1.915	
	$\delta = 1.7$	2.203	0.956	1.551	
	$\delta = 1.9$	1.915	0.831	1.348	
Tsallis	$\eta = 1.1$	0.743	0.340	0.548	Decreasing
	$\eta = 1.5$	0.322	0.168	0.267	
	$\eta = 1.7$	0.180	0.096	0.154	
	$\eta = 1.9$	0.057	0.031	0.051	
Havrda and Charvat	$\omega = 1.1$	0.004	0.002	0.004	Increasing
	$\omega = 1.5$	0.176	0.095	0.162	
	$\omega = 1.7$	0.433	0.232	0.405	
	$\omega = 1.9$	0.899	0.471	0.847	
Arimoto	$\varepsilon = 1.1$	0.008	0.003	0.005	Increasing
	$\varepsilon = 1.5$	0.652	0.161	0.281	
	$\varepsilon = 1.7$	3.329	0.454	0.824	
	$\varepsilon = 1.9$	177.609	1.104	2.115	
Boekee and Lubba	$\tau = 1.1$	0.756	0.344	0.556	Increasing Decreasing
	$\tau = 1.5$	0.466	0.233	0.371	
	$\tau = 1.7$	0.394	0.202	0.321	
	$\tau = 1.9$	0.343	0.177	0.282	
Mathai and Haubold	$\zeta = 1.1$	-0.517	-0.283	-0.455	Increasing Decreasing
	$\zeta = 1.5$	-0.322	-0.167	-0.266	
	$\zeta = 1.7$	-0.214	-0.106	-0.169	
	$\zeta = 1.9$	-0.083	-0.037	-0.061	

Estimation

In this section, we estimate the parameters of the ML-II by following the method of MLE as this method provides the maximum information about the unknown model parameters. Let $X_1, X_2, X_3, \dots, X_n$ be a random sample of size n from the ML-II, then the likelihood function $L(\theta) = \prod_{i=1}^n f(x; \alpha, \beta)$ of X is given by

$$L_{ML-II}(\theta) = (\beta(\alpha + 1))^n \prod_{i=1}^n \frac{(1 - x_i)^{\beta-1}}{(1 + \alpha x_i)^{\beta+1}}$$

The log-likelihood function $\text{Log}L(\theta) = l(\theta)$ of X is

$$l_{ML-II}(\theta) = n(\log\beta + \log(1 + \alpha)) + (\beta - 1) \sum_{i=1}^n \log(1 - x_i) - (\beta + 1) \sum_{i=1}^n \log(1 + \alpha x_i) \quad (12)$$

Now, we are concerned about obtaining the MLEs of the ML-II. For this, first, we maximize the Equation (12) and second, we calculate the partial derivatives *w.r.t.*, the unknown parameters (α, β) and equate to zero, respectively. The score vector components are given by

$$U(\theta) = \frac{\partial l}{\partial \theta} = \left(\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta} \right)^T$$

Partial derivatives *w.r.t.* α, β are given as follows, respectively

$$\frac{\partial l_{ML-II}}{\partial \alpha} = \frac{n}{1+\alpha} - (\beta+1) \sum_{i=0}^n \left(\frac{x_i}{1+\alpha x_i} \right)$$

$$\frac{\partial l_{ML-II}}{\partial \beta} = \frac{n}{\beta} + \sum_{i=0}^n \log(1-x_i) - \sum_{i=0}^n \log(1+\alpha x_i)$$

The last two non-linear equations do not provide the analytical solution for the MLEs and the optimum value of α , and β . The Newton-Raphson (an appropriate) algorithm plays a supportive role in these kinds of ML estimates. Numerical solutions and estimates of the parameters are calculated using R version 3.6.2 (statistical software).

Simulation

In this sub-section, to observe the asymptotic performance of MLE's $\hat{\varphi} = (\hat{\alpha}, \hat{\beta})$, we discuss the following algorithm.

Step - 1: A random sample $x_1, x_2, x_3, \dots, x_n$ of sizes $n = 150, 200, 250, 300, 350, 400, 450$, and 500 from Equation (5).

Step - 2: The required results are obtained based on the different combinations of the model parameters placed in S-IX ($\alpha = 3.2, \beta = 2.5$), S-X ($\alpha = 0.2, \beta = 0.9$), and S-XI ($\alpha = 0.2, \beta = 1.5$).

Step - 3: Results of mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) are calculated via *nlmib* in R. These results are presented in Tables 3 to 8.

Step - 4: Each sample is replicated 1000 times.

Step - 5: A gradual decrease with the increase in sample size is observed in mean, biases, MSEs, and Var.

Step - 6: CPs of all the parameters $\varphi = (\alpha, \beta)$ are approximately 0.975, approaching the nominal value, and AW decreases when sample sizes increases.

Furthermore, the frequent use of the measures in the development of average estimate (AE), bias, MSE, CP, AW, are given as follows:

$$AE(\hat{\varphi}) = \frac{1}{N} \sum_{i=1}^N \hat{\varphi}, \text{Var}(\hat{\varphi}) = \frac{1}{N} \sum_{i=1}^N (\varphi - \bar{\varphi}_i)^2, \text{Bias}(\hat{\varphi}) = \frac{1}{N} \sum_{i=1}^N (\hat{\varphi}_i - \varphi)$$

$$MSE(\hat{\varphi}) = \frac{1}{N} \sum_{i=1}^N (\hat{\varphi}_i - \varphi)^2, CP(\hat{\varphi}) = \sum_{i=1}^N I(\hat{\varphi}_i - 0.975 se_{\hat{\varphi}}, \hat{\varphi}_i + 0.975 se_{\hat{\varphi}}), \text{ and}$$

$$AW(\hat{\varphi}) = \frac{1}{N} \sum_{i=1}^N |(\hat{\varphi}_i + 0.975) - (\hat{\varphi}_i - 0.975)|$$

Table 3. Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of (α) for S-IX.

Sample	Mean	Var.	Bias	MSE	CP	AW
150	3.126	2.822	-0.074	1.681	0.912	6.383
200	3.157	2.262	-0.043	1.505	0.915	5.527
250	3.163	1.823	-0.036	1.351	0.915	4.931
300	3.142	1.442	-0.058	1.202	0.919	4.469
350	3.155	1.211	-0.045	1.101	0.928	4.137
400	3.155	1.109	-0.046	1.054	0.922	3.867
450	3.155	0.956	-0.045	0.979	0.929	3.942
500	3.174	0.843	-0.026	0.919	0.929	3.463

Table 4. Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of (β) for S-IX.

Sample	Mean	Var.	Bias	MSE	CP	AW
150	2.964	1.664	0.464	1.370	0.950	4.491
200	2.835	1.066	0.334	1.085	0.936	3.397
250	2.759	0.743	0.259	0.900	0.942	2.809
300	2.713	0.520	0.213	0.752	0.956	2.431
350	2.669	0.382	0.169	0.641	0.951	2.148
400	2.652	0.319	0.152	0.585	0.950	1.974
450	2.632	0.257	0.133	0.524	0.954	1.823
500	2.607	0.209	0.107	0.471	0.953	1.693

Table 5. Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of (α) for S-X.

Sample	Mean	Var.	Bias	MSE	CP	AW
150	0.255	0.083	0.055	0.293	1.000	1.432
200	0.247	0.069	0.048	0.267	0.998	1.218
250	0.236	0.058	0.036	0.243	0.996	1.070
300	0.223	0.046	0.023	0.216	0.997	0.962
350	0.218	0.040	0.018	0.201	0.996	0.883
400	0.216	0.037	0.015	0.193	0.990	0.824
450	0.211	0.033	0.011	0.182	0.994	0.772
500	0.211	0.031	0.011	0.175	0.992	0.730

Table 6. Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of (β) for S-X.

Sample	Mean	Var.	Bias	RMSE	CP	AW
150	0.906	0.013	0.006	0.115	0.970	0.587
200	0.904	0.011	0.004	0.104	0.954	0.499
250	0.906	0.009	0.006	0.095	0.957	0.442
300	0.906	0.008	0.006	0.088	0.969	0.401
350	0.906	0.007	0.006	0.081	0.969	0.369
400	0.905	0.006	0.005	0.078	0.968	0.345
450	0.906	0.005	0.006	0.075	0.971	0.324
500	0.9050	0.005	0.005	0.072	0.960	0.306

Application

In this section, we explore the application of ML-II in medical science and engineering data. For this, we consider two lifetime datasets. The first dataset was originally reported by Smithson and Verkuilen.³⁶ They studied a group of 166 healthy women's anxiety performance, i.e., outside of a pathological clinical setting, from Townsville, Queensland, Australia; data points are 0.01, 0.17, 0.01, 0.05, 0.09, 0.41, 0.05, 0.01, 0.13, 0.01, 0.05, 0.17, 0.01, 0.09, 0.01, 0.05, 0.09, 0.09, 0.05, 0.01, 0.01, 0.01, 0.29, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.09, 0.37, 0.05, 0.01, 0.05, 0.29, 0.09,

Table 7. Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of (α) for S-XI.

Sample	Mean	Var.	Bias	MSE	CP	AW
150	0.263	0.097	0.064	0.319	1.00	1.633
200	0.254	0.082	0.054	0.291	0.998	1.382
250	0.243	0.068	0.043	0.264	0.995	1.211
300	0.228	0.055	0.028	0.235	0.998	1.087
350	0.222	0.047	0.022	0.219	0.996	0.996
400	0.219	0.044	0.019	0.211	0.994	0.928
450	0.214	0.039	0.014	0.198	0.993	0.869
500	0.213	0.036	0.013	0.191	0.988	0.821

Table 8. Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of (β) for S-XI.

Sample	Mean	Var.	Bias	MSE	CP	AW
150	1.509	0.052	0.009	0.227	0.957	1.335
200	1.508	0.045	0.008	0.208	0.945	1.125
250	1.511	0.037	0.011	0.194	0.954	0.992
300	1.513	0.032	0.013	0.180	0.963	0.898
350	1.513	0.028	0.013	0.168	0.965	0.824
400	1.512	0.026	0.013	0.163	0.960	0.768
450	1.514	0.024	0.015	0.156	0.966	0.722
500	1.512	0.022	0.012	0.149	0.958	0.680

0.01, 0.25, 0.01, 0.09, 0.01, 0.05, 0.21, 0.01, 0.01, 0.01, 0.13, 0.17, 0.37, 0.01, 0.01, 0.09, 0.57, 0.01, 0.01, 0.13, 0.05, 0.01, 0.01, 0.01, 0.01, 0.09, 0.13, 0.01, 0.01, 0.09, 0.09, 0.37, 0.01, 0.05, 0.01, 0.01, 0.13, 0.01, 0.57, 0.01, 0.01, 0.09, 0.01, 0.01, 0.01, 0.01, 0.01, 0.05, 0.01, 0.01, 0.01, 0.13, 0.01, 0.25, 0.01, 0.01, 0.09, 0.13, 0.01, 0.01, 0.05, 0.13, 0.01, 0.09, 0.01, 0.05, 0.01, 0.05, 0.01, 0.09, 0.01, 0.37, 0.25, 0.05, 0.05, 0.25, 0.05, 0.05, 0.01, 0.05, 0.01, 0.01, 0.01, 0.17, 0.29, 0.57, 0.01, 0.05, 0.01, 0.09, 0.01, 0.09, 0.49, 0.45, 0.01, 0.01, 0.01, 0.05, 0.01, 0.17, 0.01, 0.13, 0.01, 0.21, 0.13, 0.01, 0.01, 0.17, 0.01, 0.01, 0.21, 0.13, 0.69, 0.25, 0.01, 0.01, 0.09, 0.13, 0.01, 0.05, 0.01, 0.01, 0.29, 0.25, 0.49, 0.01, 0.01.

The second dataset was recently reported by Rahman *et al.*³⁷ They studied the lifetime (in days) of 30 electronic devices and the dataset was as follows: 0.020, 0.029, 0.034, 0.044, 0.057, 0.096, 0.106, 0.139, 0.156, 0.164, 0.167, 0.177, 0.250, 0.326, 0.406, 0.607, 0.650, 0.672, 0.676, 0.736, 0.817, 0.838, 0.910, 0.931, 0.946, 0.953, 0.961, 0.981, 0.982, 0.990.

The ML-II is compared with its competing models (presented in Table 9) based on a series of criteria, namely, -Log-likelihood (-LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and Kolmogorov Smirnov test (K-S) test statistics. Table 10 presents a set of descriptive statistics and Tables 11 and 12 present the parameter estimates and standard errors (in parenthesis) along the goodness-of-fit test. As seen in Tables 11 and 12, the performance of the ML-II abundantly satisfies the criteria of a better fit model. Consequently, we declare that the ML-II is a better fit among all competing models on the anxiety in women data. Moreover, Figures 5 and 6 present the empirically fitted plots comprising PDF (a), CDF (b), Kaplan-Meier Survival (c), Probability-Probability (P-P) (d), Box (e), and total test time (TTT) (f) plots (see Aarset³⁸), which confirm the close fit to the data as well.

Table 9. Competitive models Cumulative distribution functions (cdfs). Kum = Kumaraswamy distribution ; L-I = Lehmann type I distribution; L-II = Lehmann type II distribution ; T-Kum = transmuted Kumaraswamy distribution ; Mostafa type-II distribution ; T-GPW = transmuted generalized power Weibull distribution; T-Lin = transmuted Lindley distribution.

Abbreviations	Model	Parameter/ variable range	Reference
Kum	$G_I(x; \alpha, \beta) = 1 - (1 - x^\alpha)^\beta$	$\alpha, \beta > 0,$ $0 < x < 1$	Kumaraswamy ³⁹
L-I	$G_{II}(x; \alpha) = x^\alpha$	$\alpha > 0,$ $0 < x < 1$	Lehmann ¹
L-II	$G_{III}(x; \alpha) = 1 - (1 - x)^\alpha$	$\alpha > 0,$ $0 < x < 1$	
T-Kum	$G_{IV}(x; \alpha, \beta, \varepsilon) = (1 + \varepsilon) \left(1 - (1 - x^\alpha)^\beta \right) - \varepsilon \left(1 - (1 - x^\alpha)^\beta \right)^2$	$\alpha, \beta > 0,$ $ \varepsilon \leq 1$ $x > 0$	Khan et al. ⁴⁰
MT-II	$G_V(x; \alpha) = \exp(x^\alpha \log 2) - 1$	$\alpha > 0,$ $0 < x < 1$	Muhammad ⁴¹
T-GPW	$G_{VI}(x; \alpha, \beta, \varepsilon) = (1 + \varepsilon) \left(1 - e^{(1 - (1 + x^\alpha))^\beta} \right) - \varepsilon \left(1 - e^{(1 - (1 + x^\alpha))^\beta} \right)^2$	$\alpha, \beta > 0,$ $ \varepsilon \leq 1,$ $x > 0$	Khan ⁴²
T-Lin	$G_{VII}(x; \alpha) = (1 + \varepsilon) \left(1 - \frac{1 + \alpha + \alpha x}{1 + \alpha} e^{(-\alpha x)} \right) - \varepsilon \left(1 - \frac{1 + \alpha + \alpha x}{1 + \alpha} e^{(-\alpha x)} \right)^2,$	$\alpha > 0$ $ \varepsilon \leq 1,$ $x > 0$	Khan et al. ⁴³

Table 10. Descriptive statistics.

Data	Minimum	1 st Quart	Mean	3 rd Quart	Skewness	Kurtosis	95% C.I.	Maximum
Women's anxiety	0.010	0.010	0.091	0.130	2.212	7.959	(0.071, 0.112)	0.690
Electronic devices	0.020	0.143	0.494	0.892	0.062	1.312	(0.353, 0.634)	0.990

Table 11. Maximum likelihood estimations with standard errors (in parenthesis) and goodness of fit for the women's anxiety data.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\varepsilon}$	-LL	AIC	BIC	HQIC	K-S
ML-II	27.793 (7.857)	0.988 (0.154)	—	-258.657	-513.315	-507.091	-510.788	0.276
T-GPW	0.780 (0.046)	5.898 (1.005)	0.453 (0.213)	-250.605	-495.210	-485.874	-491.420	0.292
T-Kum	0.703 (0.048)	3.805 (0.688)	0.569 (0.178)	-247.915	-489.831	-480.495	-486.041	0.290
Kum	0.642 (0.045)	4.473 (0.582)	—	-244.314	-484.629	-478.405	-482.103	0.287
T-Lin	9.434 (0.960)	—	0.555 (0.124)	-239.783	-475.565	-469.341	-473.039	0.376
L-II	9.082 (0.705)	—	—	-218.538	-435.077	-431.965	-433.814	0.412
L-I	0.299 (0.023)	—	—	-187.727	-373.454	-370.342	-372.191	0.270

Table 12. Maximum likelihood estimations (MLEs) with standard errors (in parenthesis) and goodness of fit for the lifetime (in days) of 30 electronic devices data.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\varepsilon}$	-LL	AIC	BIC	HQIC	K-S
ML-II	4.852 (3.842)	0.428 (0.115)	—	-4.401	-4.802	-2.000	-3.906	0.128
T-Kum	0.609 (0.178)	0.585 (0.174)	0.125 (0.495)	-3.534	-1.069	3.134	0.276	0.409
Kum	0.587 (0.161)	0.611 (0.134)	—	-3.502	-3.005	-0.203	-2.108	0.385
L-II	0.780 (0.142)	—	—	-1.00	-0.007	1.393	0.440	0.029
L-I	0.816 (0.149)	—	—	-0.659	0.680	2.081	1.128	0.189
MT-II	0.700 (0.139)	—	—	-1.056	-0.113	1.287	0.334	0.253

Figure 5. Fitted Probability density function (PDF) (a), Cumulative distribution function (CDF) (b), Kaplan-Meier survival (c), Probability – Probability (PP) (d), Box (e), and Total Test Time (TTT) (f) Plots for women's anxiety data.

Figure 6. Fitted Probability density function (a), Cumulative distribution function (b), Kaplan-Meier survival (c), Probability-Probability (d), Box (e), and Total Test Time (f) Plots for 30 electronic devices data.

Conclusion

We developed a potentiated lifetime model that exhibited the bathtub-shaped failure rate and addressed the most efficient and consistent results over complex random phenomena in this article. We called the proposed model a modified Lehmann type II (ML-II) model because it was a modified version of the Lehmann type II model. Several structural and reliability measures were developed and discussed. We used the maximum likelihood estimation method to estimate model parameters, and we also ran a simulation study to investigate the asymptotic performance of the MLEs. Data on anxiety in women from Smithson and Verkuilen³⁶ and data on 30 electronic devices from Rahman *et al.*,³⁷ were modeled to reveal the dominance ML-II over its competitors. It is hoped that in the future, the ML-II will be regarded as a superior alternative to the baseline model.

Data availability

Figshare: two_dataset.doc, <https://doi.org/10.6084/m9.figshare.14903142>.⁴⁵

This project contains the following extended data:

- Smithson and Verkuilen dataset: anxiety test scores from a group of 166 healthy women
- Rahman *et al.*, dataset: lifetime (in days) of 30 electronic devices

The authors of this work sought and obtained consent to use the dataset from the respective authors. The dataset from Smithson and Verkuilen³⁷ was originally published on [Smithson's website](#) and saved under Example 2.text. Rahman *et al.*'s dataset is available in their publication.³⁸ The dataset was published in an open repository with the authors' permission.

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Competing interests

No competing interests were disclosed.

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1. The linear representations of CDF and PDF are based on some infinite sums. The authors need to prove that these sums converge for all or some values of the parameters.
2. In page 6, it is better to present the proofs of the propositions 1 and 2.
3. In page 6, is it possible to give mathematical proof about the shapes of the pdf and hazard rate function?
4. In pages 7-8, what are the ranges of the skewness and kurtosis for the new model?
5. In order to check the accuracy of the expressions of the r th moments, the authors need to compare the obtained results with the numerical results based on Monte Carlo approximations.
6. The authors need to show that the introduced distribution is identifiable with respect to parameter vector $\vartheta=(\alpha,\beta)^T$.
7. The authors discussed only one estimation methods. Can they attempt others estimation methods and compare their performances?
8. The description of the estimation procedure is a little bit too brief. It would be good to expand on the algorithm used to estimate the model parameters (are they using a Newton-Raphson algorithm with the exact Hessian matrix?).
9. The likelihood equations are given in page 15. Obviously, they have not got an explicit solution and the MLE must be numerically calculated. At this point, the authors should do a study (theoretical or at least numerical): have these equations always got a solution? if so, is it unique? In this context, the authors must give more details about this issue, considering some plots of the log-likelihood (profile log-likelihood) function.

10. Confidence intervals for the parameters were based on asymptotic normality. In page 15, the authors should add the asymptotic normality of $\vartheta=(\alpha,\beta)^T$. Also, the authors should add the interval estimates for ϑ in real data sets.
11. At the end of Section (Application), more information about the numerical computation will be useful. For example, does the procedure involve iteration and/or initialization? What statistical package was used to obtain the results in this paper?
12. The comparison between the AIC, BIC and CAIC are important but not sufficient to decide between one or another distribution. The authors should use the Cramer-von Mises and Anderson-Darling statistics with their p-values to decide which is the best distribution.
13. A standard tool to evaluate the fit of a distribution is the P-P plot. Add this plot, in the application section.
14. In practice, much lifetime data is subject to censoring, but there is no discussion of this.
15. At least a short discussion on the estimation under Bayesian paradigm would be really nice. This would really strengthen the content of the paper.

Is the work clearly and accurately presented and does it cite the current literature?

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Yes

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Are the conclusions drawn adequately supported by the results?

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